A Fundamental Theorem in Synthesis of Active Balanced RC Networks and a New Realization Procedure

In the paper is presented a new proof of the fundamental theorem in network synthesis of arbitrary, square (stable) admittance matrix of real, rational functions in complex frequency $s$ (RRF matrix), by active, multiport, transformerless, balanced RC network. Since the theorem establishes only the necessary condition for network existence, it is proved firstly that this condition is also sufficient if active elements used in realization are balanced voltage-controlled voltage sources (VCVS) and then is given a formulation of a new realization procedure of RRF admittance matrices, by using theoretically minimum number of balanced VCVS. The proposed procedure relies on a new theorem on representation of regular, polynomial matrices ($\lambda$-matrices) in $s$ (having the specified degree), as product of two $\lambda$-matrices. The obtained results are the most general in nature and are easily applicable in active, transformerless, multiport, RC network synthesis.

**Keywords:** Active RC synthesis, balanced networks, transformerless synthesis, factorization of polynomial matrices.

1. INTRODUCTION

It is well-known that the admittance matrices of passive, multiport, transformerless RLC networks are paramount $[1, 2] \forall s \in [0, \infty) \ (s \text{ is, in general, the complex frequency})$. A symmetric, $P$-th order real matrix is said to be paramount if each of its $r$-th order main minors $(r=1, 2, \ldots, P)$ is not less than the absolute value of any other minor established from the same rows (columns). Since the paramountcy is a necessary condition, Telegen has tried (and succeeded) to prove that it is also sufficient for existence of two- and three-port, purely resistive networks. If $P > 3$ the paramounty does not assure the resistive $P$-port existence and the synthesis problem appears to be equivalent to synthesis of resistive network with internal nodes. Unfortunately, this problem is not solved yet, except for ladder one- and two-ports. The procedures proposed for realization of admittance matrices during past few decades reduce to active network synthesis of pertinent admittance matrices by networks with no internal nodes $[3-5]$. In these realizations the passive, transformerless, RC subnetworks are either balanced ($PBRCT_0$) $[3, 4]$, or common-ground ($PGRCT_0$) $[5]$, depending on whether the coefficient matrices of the second Foster's expansion of pertinent admittance matrix are dominant (DMCSFE) $[2-4]$, or hyperdominant (HDMCSFE), respectively $[5-9]$.

Multiport network synthesis problem is encountered yet in case of minimal synthesis of RRF, $w(s)=P(s)/Q(s)$ [$P(s)$ and $Q(s)$ are mutually prime polynomials], since it appears to be equivalent to synthesis of passive or active resistive network with $\delta w+1$ ports $\{\delta w \text{ is the degree of } w(s) \text{ in Duffin-Hazony's sense} [10]\}$. It has been proved that $\delta w=\max \{P^0, Q^0\}$, where $P^0$ and $Q^0$ are the algebraic degrees of $P(s)$ and $Q(s)$, respectively $[10, 11]$.

It has also been known $[4]$ that arbitrary stable RRF admittance matrix of order $P$ can always be realized by active $BRCT_0$ network with $P$ negative immitance converters (NICs), or by common-ground ($GRCT_0$) network with $2P$ VCVS $[5]$. In all cases $[3-5]$ the realization network and the complete set of equivalent realizations are strongly dependent on a special factorization of the $P$-th order, generally regular, polynomial matrix ($\lambda$-matrix, $\lambda_M$; "lambda matrix") in $s$. A $\lambda$-matrix is said to be generally regular if it is not singular $\forall s$. In section 2 we shall first prove our prerequisite,

**Theorem 1:** The sufficient condition for factorization of $P$-th order, generally regular $\lambda M \ P(s)$, having degree $L$, whose determinant has $K$ distinct zeros, in form $P(s)=P_0(s)\cdot P_1(s)$, where $p_2>P_1^0$ reads: $K > (P-1)L+p_2-1$. The coefficients matrices of $s$ in $P_0(s)$ and $P_1(s)$ are real or complex if the zeros of det $P_0(s)$ are real or complex, respectively.

and in sections 3 and 4 we give proof of our main result,

**Theorem 2:** For realization of arbitrary $P$-th order matrix of RRFs in complex frequency $s$, as the admittance matrix of active $BRCT_0$-$P$-port network:

(a) $P$ controlled sources (CS) with real-valued controlling coefficients are necessary,
In general, the realization can not be achieved by active P-port with less than P CS.
(c) P balanced VCVS with real-valued and by moduli greater than unity controlling coefficients (i.e. voltage "amplifications") are necessary and sufficient in general case.

2. FACTORIZATION OF REGULAR $\lambda$–MATRIX

In this section we shall prove Theorem 1. Let us first suppose that $P_{\lambda(s)}$ is a P-th order, generally regular $\lambda M$, such that $L_{\lambda(s)} = \max \bigl\{ |P_{\lambda(s)}(s)| \ (i = 1, 2, \ldots, P) \bigl\}$. Assume that the coefficient matrix of $s^k$ in $P(s)$ is regular. Then:
\[
\det P(s) = \sum_{k=0}^{P} a_k \cdot s^k. \tag{1}
\]
Suppose $\det P(s)$ has $K$ distinct zeros $s_k \ (k = 1, 2, \ldots K)$. For each $s_j = s_k$ the columns $P^{(j)}(s_k)$ ($j = 1, 2, \ldots, P$) of $P(s)$ are linearly dependent, i.e. there must exist $P$ numbers $q_{1k}, q_{2k}, \ldots, q_{Pk}$ — provided that at least one of them is different from zero, which assure the following:
\[
\sum_{j=1}^{P} q_{jk} \cdot P^{(j)}(s_k) = 0_{P,1}, \tag{2}
\]
where $0_{P,1}$ is $P$-dimensional column-vector. Let the polynomial $C(s)$ be the greatest common-divisor of all cofactors $D_{jk}(s)$ ($i = 1, 2, \ldots, P$) of $P(s)$. Laplace expansion of $\det P(s)$ with respect to entries of the $k$-th column of $P(s)$ reads:
\[
\det P(s) = \sum_{j=1}^{P} p_{jk} \cdot D_{jk}(s) = C(s) \sum_{j=1}^{P} p_{jk} \cdot D_{jk}(s). \tag{3}
\]
If $s = s_k$ is a zero of $C(s)$, then rank $P(s_k) = P - 1$. If polynomial $\det P(s)/C(s)$ has at least one zero $s = s_k$ which is different from zeros of $C(s)$ and furthermore rank $P(s) = P - 1$, then the following systems of simultaneous linear algebraic equations in $q_{jk} \ (j = 1, 2, \ldots, P)$ have only the trivial solution and a non-trivial one, respectively:
\[
\sum_{j=1}^{P} q_{jk} \cdot P^{(j)}(s_k) = 0_{P,1} ; \quad \sum_{j=1}^{P} q_{jk} \cdot P^{(j)}(s_k) = 0_{P,1}. \tag{4}
\]
Since $P(s)$ is generally regular, we can put down:
\[
\sum_{j=1}^{P} q_{jk} \cdot P^{(j)}(s_k) = (s - s_k) \cdot M^{(k)}(s), \tag{5}
\]
where $M^{(k)}(s)$ is a $P$-dimensional $\lambda$-column-vector with degree not greater than $1$. Now, let us consider only those zeros $s = s_k$ of $\det P(s)$ which provide rank $P(s_k) = P - 1$. Then, at least one set of different indices (i.e. rows) $\{i_1, i_2, \ldots, i_{P-1}\}$ must exist for which holds:
\[
\sum_{j=1}^{P} q_{jk} \cdot P^{(j)}(s_k) = -q_{kk} \cdot P^{(k)}(s_k), \tag{6}
\]
Since the second system of (4) has a non-trivial solution, we can first select $q_{kk} \neq 0$ arbitrarily and then we may calculate $q_{jk} \ (j \neq k, j = 1, 2, \ldots, P)$ uniquely from (6).

From assumption that rank $P(s_k) = P - 1$ it follows that a non-zero minor of $P(s_k)$ reads:
\[
\det \begin{bmatrix} P^{(1)}(s_k) & \ldots & P^{(P-1)}(s_k) \\ \vdots & \ddots & \vdots \\ q_{1k} & \ldots & q_{Pk} \end{bmatrix}, \tag{7}
\]
and that it provides the existence of the constant, regular matrices $Q_k \ (k = 1, 2, \ldots, P)$:
\[
Q_k = \begin{bmatrix} q_{1k} & \ldots & q_{Pk} \end{bmatrix}, \tag{8}
\]
which enable the following representation of $P(s)$:
\[
P(s) \cdot Q_k = \begin{bmatrix} P^{(1)}(s) & \ldots & P^{(P-1)}(s) \\ \vdots & \ddots & \vdots \\ P^{(P)}(s) \end{bmatrix}, \begin{bmatrix} (s - s_k) \cdot M^{(k)}(s) \end{bmatrix}, \tag{9}
\]
Since $C_k \ (P-1)$, number of distinct zeros of $C_k(s)$ is not greater than $(P-1)-L$. The sufficient condition for any factorization of $P(s)$ reads $K > (P-1)-L$, since only then can exist at least one zero of polynomial det $P(s)/C_k(s)$, which is different from zeros of $C_k(s)$. Let this condition be satisfied and let there exists at least $P$ distinct zeros $s_1, s_2, \ldots, s_P$ of det $P(s)$. If $L_1 > 1(L-1)$ the existence of these zeros is automatically assured if $K > (P-1)-L$. Then the following factorization of $P(s)$ holds:
\[
P(s) = P(s) \cdot Q_1 \cdot Q_1^{-1} = \begin{bmatrix} (s - s_1) \cdot M^{(1)}(s) \\ \vdots \\ (s - s_P) \cdot M^{(P)}(s) \end{bmatrix}, \begin{bmatrix} (s - s_1) \cdot M^{(1)}(s) \\ \vdots \\ (s - s_P) \cdot M^{(P)}(s) \end{bmatrix}, \begin{bmatrix} Q_1^{-1} \cdot Q_1 \end{bmatrix}, \begin{bmatrix} Q_1^{-1} \cdot Q_1 \end{bmatrix}^{T} \tag{10}
\]
where $Q_1 = \begin{bmatrix} Q_1 & Q_2 & \ldots & Q_P \end{bmatrix}$, and $D_1 = \begin{bmatrix} (s - s_1) \cdot (s - s_2) \cdot \ldots \cdot (s - s_P) \end{bmatrix}$.

$Q_1$ is a regular, constant, P-order matrix. In this way, the generally regular matrix $P(s)$ is represented as product of two regular, P-order $\lambda$-matrices: $N_1 = \begin{bmatrix} M^{(1)}(s) \\ \vdots \\ M^{(P)}(s) \end{bmatrix}$ with polynomial degree $L-1$ and $T(s) = D_1(s)$ with unit polynomial degree. By extraction of a linear matrix factor $T_1(s)$, whose determinant is a P-th order polynomial in $s$, we have achieved that the number of distinct zeros of det $N_1$, becomes equal to $K=K-P$.

Let us suppose that $s = s_k$ is zero of det $P(s)$ having order $t_k > 1$ and, in addition, that the rank $P(s_k) > R - 1$. It is convenient that first $r$ columns of $P(s_k)$ be linearly independent. Since this is not the case, in general, we must apply the right-hand multiplication of $P(s_k)$ by a transposition matrix $R = R_1 \cdot R_2 \cdot \ldots \cdot R_r$ having regular, column-permutation matrix factors. Permutation matrix of the $i$-th and $j$-th column of $P(s_k)$ is regular, right-hand matrix multiplicant of $P(s_k)$, whose entries are:
\[
r_{mn} = \delta_{mn}, \ (m,n) \neq (i,j); \quad r_{ij} = r_{ji} = 0, \quad r_{ij} = r_{ji} = 1. \tag{11}
\]
where $\delta_{mn}$ is Kronecker's $\delta$-symbol. In this way, we can always represent $P(s)$ as $P(s) = R^{-1}\left[ P^{(1)} \right] P^{(2)} \cdots P^{(r)}$ where the first $r$ columns of $[P^{(r)} \cdot \cdots \cdot P^{(1)}]$ are linearly independent. Bearing this in mind, we infer that the system of linear equations in $q_{jk}$:

$$\sum_{j=1}^{r} q_{jk} P^{(j)}(s_{k}) = 0_{r,1},$$

(12)

has only the trivial solution since determinant of system,

$$\sum_{j=1}^{r} q_{jk} P^{(j)}(s_{k}) = 0_{r,1}$$

(13)

with maximum order $(r < P)$ is different from zero.

Let us now construct the following set of constant, regular matrices $\{Q^{(1)}, Q^{(2)}, \ldots, Q^{(P-r)}\}$ so as to be:

$$P(s) \cdot R \cdot Q^{(1)} \cdots Q^{(P-r)} = \left[ P^{(1)} \mid \ldots \mid P^{(r)} \right] \cdot \left[ \sum_{j=1}^{r} q_{i,j} P^{(j)} \right],$$

(14)

$$Q^{(j)} = \left[ q_{i,j}^{(j)} \right]_{p,p} \left[ \begin{array}{c|c|c} 1 & 1 & \ldots \end{array} \right] = \left[ q_{i,j}^{(j)} \right]_{p,p} \left[ \begin{array}{c|c|c} 1 & 1 & \ldots \end{array} \right] \left( 1 \right),$$

(15)

Matrices $Q^{(j)}$ are regular if and only if for $\forall j = 1, 2, \ldots, (P-r)$, $q_{i,j}^{(j)} \neq 0$. These entries are to be assumed arbitrarily, provided that the previous condition is satisfied.

Since rank $(P-R) = r$, then for $s = s_{k}$ the following systems of linear equations in $q_{i,j}^{(j)} \left[ i=1, 2, \ldots, (r+1) \right]$ have unique solutions for $\forall j = 1, 2, \ldots, (P-r)$ [recall that $q_{i,j}^{(j)} \neq 0$ have already been arbitrarily determined]:

$$\sum_{j=1}^{r+1} q_{i,j}^{(j)} P^{(j)}(s_{k}) = (s-s_{k})M^{(r+1)}(s).$$

(16)

Once the matrices $Q^{(j)}$ are known, then from (16) matrices $M^{(r+1)}(s)$ can easily be determined. Thereof we get:

$$P(s) = \left[ P(s) \cdot R \cdot Q^{(1)} \cdots Q^{(P-r)} \right] P^{(2)} \cdots P^{(1)} = \left[ P^{(1)} \mid \ldots \mid P^{(r)} \right] \left[ \begin{array}{c|c|c|c} 1 & 1 & \ldots \end{array} \right]$$

(17)

where $Q^{(1)}, Q^{(2)}, \ldots, Q^{(P-r)}$. In this way we have proved that if rank $P(s) = r < P - 1$ matrix $P(s)$ can be represented as product of two matrices regular for $s \neq s_{k}$. The first of them, $N_{L-1} = \left[ P^{(1)} \mid P^{(2)} \ldots P^{(r)} \right] \left[ M^{(r+1)} \mid \ldots \mid M^{(P-r)} \right]$ has degree $L$ or $L-1$, whereas the second one, $T_{r}(s) = D(s) \cdot R^{-1} \cdot Q^{(1) \cdot \cdots \cdot Q^{(P-r)}}$ has unit-degree, where $D(s)$ is diagonal matrix $D_{r}(s) = \text{diag} [1 \ldots 1 (s-s_{1}) \ldots (s-s_{1}) \ldots (s-s_{1})]$, $(r-1)$ unities.

Up to now we have proved that the condition $K > (P-1)L$ assures extraction of linear $\lambda$-matrix-factor from $P(s)$. In the sequel we shall derive a sufficient condition for factorization of a regular $\lambda$-matrix $P(s) = P^{(0)}(s)$, where both $P^{(0)}(s)$ and $P^{(s)}$ are regular and, in addition, $P^{(0)} = P^{(1)} \cdots P^{(2)} \cdots P^{(r)} \cdots P^{(1)}$ are depending on the rank of $P(s_{k})$, $N_{L-1} = (s_{k})$, etc. If $P_{\min} = P_{\min} \geq 2$ we have analogously:

$$N_{L-1} = N_{L-2} \cdot T_{2}, \quad N_{L-2} = N_{L-3} \cdot T_{3}, \ldots, \quad (18)$$

where number of distinct zeros of $det N_{L} = \lambda_{n}$ is smaller than $K_{n} = K_{n} = P_{n} = P_{n} - 2$, of $det N_{L}$, $n$ is not smaller than $K_{n} = P_{n} = P_{n} - 2$, and so on. Let us now suppose that in the $(h-1)$-st step we have represented $P(s)$ as product of two $\lambda$-matrices $N_{L-1} = T_{h}$ having degrees in $[L-h+1, L]$ and $h-1$, respectively. For extraction of a linear matrix factor from $N_{L-1}$ it is sufficient that number $K_{h}$ of distinct zeros of $det N_{L-1}$ is greater than $(P-1)(L-h+1)$. Since $K_{h}$ is not smaller than $K-1 + P_{h} - P_{h}$, and taking into account the condition $L \geq h+1$, which is necessary and sufficient for $N_{L_{h}}$ to be at least monome in $s$, we obtain:

$$K - (h-1)P > (P-1)[L-(h-1)] \quad L \geq h+1. \quad (19)$$

This is the worst case sufficient condition for extraction of a linear matrix factor. Therefrom, for $h = 2$, we get:

$$K > (P-1)L + 2P_{1} - 1 \quad L \geq P_{1} + 1. \quad (20)$$

This is the sufficient condition for representation of $P(s)$ as product of two regularly general $\lambda$-matrices $P^{(0)}(s)$ and $P_{1}(s)$ with degrees $P_{1} = [L-P_{1}, L]$ and $P_{1} = [L-P_{1}]$, respectively. For selected $P_{1} = [L-P_{1}] \Rightarrow P_{1} = [L-P_{1}, L]$ depending on ranks of matrices obtained during application of the proposed procedure. It is obvious that coefficient-matrices of $s$ in $P(s)$ and $P_{1}(s)$ are real or complex if the zeros of $det P(s)$ are real or complex, respectively.

This completes the proof of Theorem 1.

**Comment:** In proving theorem, the $P$-th order coefficient matrix of $s^{P}$ in $P(s)$ is supposed to be regular. On contrary, if it is singular and has the rank $r < P$, it can be shown that sufficient condition for extraction of a linear matrix factor from $P(s)$ becomes slightly weaker than the previous condition $K > (P-1)L$ and it now reads:

$$K > (P-1)L - (P-r). \quad (21)$$

The proof of assertion (21) is straightforward and is left to the reader.

### 3. GENERAL ACTIVE NETWORK STRUCTURE

In this section we shall give proof of items (a) and (b) of Theorem 2, while in section 4 we prove item (c), by formulation of new, general synthesis procedure. At the outset consider general linear, active BT, or $BRCT_{0}$ multiport network $N_{s} = N_{s} \cup N_{p}$ depicted in Fig. 1. $N_{s}$ is linear active subnetwork, while $N_{p}$ is linear $BRLCT$ or $BRCT_{0}$ multiport subnetwork with arbitrary number of passive elements. $N_{s}$ contains arbitrary number of controlled sources (CS) of all four possible types. Let $E = [E_{1}, E_{2}, E_{3}]$ and $Y = [I_{1}, I_{2}, I_{3}]$ (*" transposition of matrix) be the $P$-dimensional column-vectors of Laplace transforms of voltages and currents at accessible ports, respectively.
Let $A = [I_1 \ldots I_n E_1 \ldots E_m]^T$ be $(h+m)$-dimensional column-vector of Laplace transforms of controlled currents and voltages of $CS$ and let $B = [I_1 \ldots I_n E_1 \ldots E_m]^T$ be $(j+k)$-dimensional column-vector of Laplace transforms of controlling currents and voltages. Denote by $C$, $(h+m) \times (j+k)$ RRF matrix whose entries are coefficients of generalized CS imbedded into subnetwork $N_A$. Certainly, it must hold $A = C \cdot B$. Assuming $E$ and $A$ to be the network excitations, then by force of superposition principle, which holds in linear electrical networks, we may write:

$$I = Y_0 \cdot E + D \cdot A \land B = F \cdot E + G \cdot A,$$

(22)

where: (i) $Y_0$ is the admittance-matrix of $P$-port network produced by elimination of $CS$ ($\Leftrightarrow A = 0$), (ii) $D$ is $P \times (h+m)$ matrix, whose entries are current-transmittances and admittances, (iii) $F$ is $(j+k) \times P$ matrix, whose entries are voltage-transmittances and admittances and (iv) $G$ is $(j+k) \times (h+m)$ matrix having as its entries the transfer-admittances, or current- and voltage-transmittances.

![Figure 1. General active realization network](image)

Let $U_{h+m}$ be the $(h+m)$-th order identity matrix. If matrix $U_{h+m} \cdot C \cdot G$ is regular, then from (22) we obtain the admittance matrix $Y = Y(s) [I - Y(s) E]$ of the overall $P$-port, active, $BRLCT$ or $BRCT_0$ network (Fig. 1), which in general may have all four types of generalized $CS$:

$$Y(s) = Y_0 + D \cdot (U_{h+m} \cdot C \cdot G)^{-1} \cdot C \cdot F.$$  

(23)

The minimum number of $CS$ which is necessary for realization of matrix $Y$ is obtained by investigating the rank of $Y - Y_0$, whose entries are $RRF$'s having poles $s_1, \ldots, s_p$, with multiplicities, say, $n_1, \ldots, n_p$, respectively. Since Laurent's complex expansion of $Y(s) - Y_0(s)$ reads:

$$Y(s) - Y_0(s) = \sum_{k=0}^{P} A_k \cdot s^k + \sum_{i=1}^{n \min (j+k, h+m)} B_{(i)}^{(k)} \cdot (s-s_i)^{-k},$$

(24)

and since the rank of product of two matrices never exceeds the ranks of matrix factors, we can write,

$$\text{rank } (Y - Y_0) \leq \text{min } (j+k, h+m),$$

$$\text{rank } C \leq \min (j+k, h+m),$$

(25)

and $s \rightarrow \infty$, $\text{rang } A_p \leq \text{rang } C$, 

$$\text{rang } B_{(i)}^{(k)} \leq \text{rang } C, \quad (i = 1, 2, \ldots, q).$$

(26)

The matrix $Y_0$ must be positive real [2, 10], since it relates to the passive ($B$)$RLCT$, or ($B$)$RCT_0$ network $N_0$. If network is purely resistive, $Y_0$ must be positive semidefinite. In the most general case the matrix $Y(s)$ can have the following two properties with respect to its poles:

1. They may lie in right-half complex plane and/or on imaginary axis having multiplicities greater than unity.
2. They may be in the left-half complex plane and are different from poles of $Y_0$, or they are the same as poles of $Y_0$, but have the greater multiplicities than poles of $Y_0$.

In both cases (1) and (2) matrix $Y_0$ does not influence the values of entries in coefficient matrices $A_p (p \geq 2)$ and $B_{(i)}^{(k)}$, $n_i > 1 (i = 1, 2, \ldots, q)$. If one of them has rank $P$, then from (25) follows rank $C \geq P$. This conclusion means that, in the most general case, for existence of active, balanced $P$-port ($B$)$RLCT$ or ($B$)$RCT_0$ network which realizes the arbitrary, $P$-th order, RRF matrix as admittance matrix of network, it is necessary that the active subnetwork $N_A$ (Fig. 1) has at least $P$ $CS$ and the same number of controlling ports. The sufficiency of the proposed conditions for existence of network realization by using negative imittance converters (NICs) as active elements is proved elsewhere [4]. That completes proof of items (a) and (b) of Theorem 2.

Unfortunately, NICs as active elements are potentially unstable. This gave us impetus to develop our realization structure and our general procedure for synthesis and realization of arbitrary admittance matrices (possibly stable). In new procedure we are going to present, the minimum number ($=P$) of balanced voltage-controlled voltage sources ($VCVS$) is implemented. Hereafter, we prove that $P$ balanced $CS$ is, in general, sufficient for realization of any $P$-th order admittance RRF matrix. The preferable matrix realizations from engineering point of view are stable ones. So, we shall apply algorithm only to those cases, bearing in mind that unstable admittance matrices can also be realized by using the same procedure. Passive subnetwork $N_P$ in realization structure proposed is balanced, transformerless and $RC$ ($BRCT_0$) realizations with no inductors ($L$) and transformers ($T$) are preferable from the practical point of view.

4. GENERAL NETWORK STRUCTURE AND A NEW REALIZATION PROCEDURE

The proposed new general active network structure is depicted in Figure 2. Observe that subnetwork $N_P$ is $BRCT_0$ and that balanced subnetwork $N_A$ contains only $VCVS$ ("voltage amplifiers"). Let us now suppose that all these amplifiers are approximately ideal. This means that their input-impedances and output-admittances are infinite and that their voltage-gains $A_1, \ldots, A_P$ are real and either finite, or infinite. Almost every operational amplifier in common use today satisfy previously stated three conditions (its open-loop gain is "infinite"). Let:

$$E = [E_1 \ldots E_P]^T, \quad I = [I_1 \ldots I_P]^T, \quad A = \text{diag}(A_1 \ldots A_P),$$

(26)

$$E_d = [E_{d1} \ldots E_{dP}]^T, \quad I_d = [I_{d1} \ldots I_{dP}]^T = 0_{P,1},$$

$$E_b = [E_{b1} \ldots E_{b3P}]^T, \quad I_b = [I_{b1} \ldots I_{b3P}]^T, \quad A = E_b A_e.$$

At this instance we recall that the admittance matrix $Y = Y(s)$ to be realized must satisfy relation $I = Y \cdot E$. On the other hand, the admittance matrix $Y(s)$ of balanced, RC transformerless, reciprocal subnetwork $N_P$ (Fig. 2) with augmented number of accessible ports ($=3P$) and no internal nodes must be symmetric and must have the
following structure [2, 10] (since network \( N_p \) is passive):

\[
Y^* = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{13} \\ Y_{12} & Y_{22} & \cdots & Y_{23} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{13} & Y_{23} & \cdots & Y_{33} \end{bmatrix} = Y \cdot \begin{bmatrix} I_a \\ I_d \end{bmatrix} = Y^* \cdot \begin{bmatrix} E_a \\ E_d \end{bmatrix}.
\]  

(27)

Bearing in mind that that always must be:

\[
I_a = [I_{p1} \ldots I_{p2}]^T = 0 \text{, } \forall p, \quad I_d = A \cdot E_d,
\]  

then by eliminating \( I_d \) and \( I_d \) from (27) and using \( I \cdot Y \cdot E \), we obtain, provided that \( Y_{22} + Y_{23} \cdot A \) is regular:

\[
Y = Y_{11} - (Y_{12} + Y_{13} \cdot A) \cdot (Y_{22} + Y_{23} \cdot A)^{-1} \cdot Y_{12}^T.
\]  

(29)

\[\text{Figure 2. The proposed network structure}\]

Our main intention in conducting the sufficiency proof of Theorem 2 is to show that for arbitrary RRF matrix \( Y(s) \) there always exist at least one (and infinite many) sets of real voltage-gains \( A_1, A_2, \ldots, A_p \) for which it is possible to identify all submatrices \( Y_{ij} \) \((i, j = 1, 2, 3)\) in (27) so as to assure dominance of coefficient matrices of the second Foster's expansion (DMCSFE) of \( Y^* \). As is well-known [10] the dominance represents necessary and sufficient condition for BRCT- realization of \( Y^* \).

Putting it sufficiently and conditionally, the VCSs play a role of "Deus ex machina" means for successful solution of all network synthesis tasks. These tasks can always be accomplished by the appropriate selection of: (i) \( P \)-th order matrix \( Y_{11} \), which must be realizable by a \( \text{BRCT}_0 \) network and (ii) finite or infinite voltage-amplifier gains.

Let \( x_p, q, N_d \) and \( D \) \((i, j = 1, 2, \ldots, P)\) be polynomials in \( s \) such that: \{1\} \( Y = [N_d]_{p,p}/D \) \((\text{\( \leftrightarrow \) } Y \) is given by statement of problem), \{2\} \( Y_{11} = [x_p]_{p,p}/q \) \((\text{\( \leftrightarrow \) } Y_{11} \) must be appropriately selected and must be realizable by \( \text{BRCT}_0 \) network) and \{3\} \( P = [D \cdot x_p-q; N_d]_{P,p} \) must be a regular, factorizable \( \lambda \)-matrix in the sense of Theorem 1. Factorizability of \( P \) can always be achieved by suitable selection of \( Y_{11} \). In addition, \( Y_{11} \) must have strictly dominant coefficient-matrices of its second Foster's expansion, although the dominance (but not strictly) of \( Y_{11} \) is necessary and sufficient for its realization by a \( \text{BRCT}_0 \) network [2, 10].

In the sequel we shall distinguish between the following two cases: (i) voltage-amplifier gains are finite and (ii) these gains are infinite. For each of these cases, we shall develop the steps of general synthesis procedure.

\[\text{Case 1*: Real and finite voltage-amplifier gains}\]

Presuming the factorizability of \( P = P_1 \cdot P_2 \), from (29) we infer that if \( c_1 \) and \( c_2 \) are arbitrarily selected reals \((c_1, c_2 \neq 0)\) — a possible identification of matrices reads:

\[
Y_{12} + Y_{13} \cdot A = \frac{P_1}{c_1 q}; \quad Y_{22} + Y_{23} \cdot A = \frac{D}{c_1 c_2 q} \cdot U_P,
\]  

(30)

\[
Y_{12} = P_2^T / c_2 q; (U_P \text{ is the } P\text{-th order identity matrix}),
\]  

wherefrom it becomes clear that \( Y_{22} + Y_{23} \cdot A \) is generally regular matrix. Let the numerical ratio \( r = c_2/c_1 \) be selected arbitrarily, wherefrom it still remains to determine \( c_2 \neq 0 \) appropriately. By virtue of \{2\}, the roots of polynomial \( q(s) \) must be simple and lie in the origin and/or on negative real axis \([2, 6, 10]\). In addition, the following must hold \( P_s \leq Q + 1 (Q = q^2) \) [2]. We can now identify both \( Y_{12} \) and \( Y_{13} \), after the Laurent's complex expansion of \( P_2^T/q^2 \):

\[
Y_{12} = \frac{P_2^T}{c_2 q}; \quad Y_{13} = \frac{1}{c_2 q} \cdot (r \cdot P_1 - P_2^T) \cdot A^{-1},
\]  

(31)

where it has been assumed \( q(s) = (s-s_1) \cdots (s-s_Q) \) and \( s_0 = 0 < s_1 < \cdots < s_Q \). By selecting sufficiently great \( c_2 \), we can produce entries of \( Y_{12} \) and \( Y_{13} \) arbitrarily small, so as to assure the true dominance of the first \( P \) rows and columns of symmetric matrix \( Y^*(s) \). Remember that \( Y^*(s) \) must be realizable by a \( \text{BRCT}_0 \) network.

All entries of matrix \( Y_{22} \) and off-diagonal entries of \( Y_{23} \) must not have poles distinct from poles of diagonal entries of \( Y_{22} \). These diagonal entries must essentially correspond to input admittances of \( \text{BRCT}_0 \) networks [2]. Let us identify now the other submatrices. We have:

\[
Y_{22} + Y_{23} \cdot A = \frac{D}{c_1 q} \cdot U_P = F_{ro} \cdot s + \sum_{m=0}^{Q} F_m \cdot \frac{s}{s + s_m},
\]  

(32)

where \( F_{ro} = [F_{ro} \cdot U_P] \) \((m = 0, 1, \ldots, Q)\) are diagonal matrices. Let \( G_{ro} = [g_{ro}^{(m)}]_{p,p} \) and \( H_{ro} = [h_{ro}^{(m)}]_{p,p} \) be arbitrarily selected diagonal matrices whose diagonal entries are non-negative and satisfy the condition \( g_{ro}^{(m)} \cdot h_{ro}^{(m)} = 0 \) \((\forall i=1, 2, \ldots, P, \forall m=\infty, 0, 1, \ldots, Q)\). Let \( D_{ro} = [d_{ro}^{(m)}]_{p,p} \) be the \( P \)-th order scalar matrix whose diagonal entries are positive and should be at first estimated and thereon assumed appropriately. If we put \( F_{ro} = (G_{ro} + D_{ro}) = (H_{ro} + D_{ro}) \), then from (32) follows a possible identification of matrices \( Y_{22} \) and \( Y_{23} \):

\[
Y_{22} = (G_{ro} + D_{ro}) \cdot s + \sum_{m=0}^{Q} (G_m + D_m) \cdot \frac{s}{s + s_m},
\]  

(33)

\[
Y_{23} = -[(H_{ro} + D_{ro}) \cdot s + \sum_{m=0}^{Q} (H_m + D_m) \cdot \frac{s}{s + s_m}] \cdot A^{-1}
\]  

Let \( K_{ro} = [k_{ro}^{(m)}]_{p,p} \) \((m = \infty, 0, 1, \ldots, Q)\). From (33) we infer that the second \( P \) rows (columns) of coefficient-matrices in the second Foster's expansion of \( Y^* \) will be dominant if \( \forall i=1, 2, \ldots, P \) and \( \forall m=\infty, 0, 1, \ldots, Q \) — the following condition holds:

\[
g_{ii}^{(m)} + d_{ii}^{(m)} \geq \sum_{j=1}^{P} \left[ g_{ij}^{(m)} + d_{ij}^{(m)} + \frac{k_{ij}^{(m)} + d_{ij}^{(m)}}{A_j} \right] + \frac{k_{ii}^{(m)} + d_{ii}^{(m)}}{A_j} + \frac{1}{c_2} \sum_{j=1}^{P} k_{ji}^{(m)}]
\]  

(34)
The previous set of conditions essentially reduces to:

\[ g_{ij}^{(m)} + d_{ij}^{(m)} \geq \frac{h_{ij}^{(m)} + d_{ij}^{(m)}}{A_{ij}} + \frac{1}{2} \sum_{j=1}^{p} k_{ij}^{(m)} \cdot (35) \]

Since \( g_{ij}^{(m)} \) is 0 (i.e., 1, ... , P; \( \forall m=\infty, 0, 1, \ldots , Q \)), dominancy of the second P rows (columns) in MCMSE of \( Y^* \) is achieved if all \( A_{ij} (i=1, 2, \ldots , P) \) are by modulus greater than unity and if all \( d^{(m)} \)-entries of selected scalar matrices, for \( m=\infty, 0, 1, \ldots , Q \) satisfy condition:

\[ d_{ij}^{(m)} \geq \max_{i=1,2, \ldots , P} \left\{ \frac{h_{ij}^{(m)}}{A_{ij}} + \frac{1}{2} \sum_{j=1}^{p} k_{ij}^{(m)} \right\} \cdot (36) \]

We can notice from (29) that \( Y \) does not depend on \( Y_{33} \). Thereof it follows that dominancy of the third \( P \) rows or columns of coefficient-matrices in the second Foster’s expansion of \( Y^* \) can always be achieved with indefinite number degrees of freedom. Among many different choices of \( Y_{33} \) the most simple one seems to be selection of \( Y_{33} \) as constant diagonal matrix with positive, sufficiently large diagonal entries.

To this end we have proved the item (c) of Theorem 2, i.e. that \( P \) balanced \( VCVs \) with finite and by modulus greater than unity voltage-gains are sufficient for realizacation of any (preferably stable) \( P \)-th order \( RRF \) matrix as admittance matrix of \( BRCT_0 \) network. It has also been proved that there exists indefinite number of topologically and parametrically equivalent network realizations.

**Case 2**: Infinite voltage-amplifier gains

According to (28), in this case it must hold \( E_{p}=0_{P,1} \) (since \( A \rightarrow \infty \)). Hence, the identification (30) will not be valid anymore. If we rearrange (29) as follows:

\[ Y = Y_{11} - (Y_{12} \cdot A^{-1} + Y_{13}) \cdot (Y_{22} \cdot A^{-1} + Y_{23})^{-1} \cdot Y_{12}^T \cdot (37) \]

and let \( A \rightarrow \infty \), we finally obtain,

\[ Y = Y_{11} - Y_{13} \cdot Y_{23}^{-1} \cdot Y_{12}^T \cdot (38) \]

By using (38) we can make the following identification of the seeked matrices:

\[ Y_{12} = \frac{P_{1}}{c_{2} \cdot q}, \quad Y_{13} = \frac{P_{1}}{c_{1} \cdot q}, \quad Y_{23} = \frac{D}{c_{1} \cdot c_{2} \cdot q} \cdot U_{p} \cdot (39) \]

In this case, all matrices \( Y_{i} (i=1, 2, 3) \) are to be selected so as to assure the dominancy of coefficient-matrices in the second Foster’s expansion of \( Y^* \).

In both cases 1º and 2º the produced matrix \( Y^* \) is to be realized by a parallel connection of at most \( q^2-1 \) \( BRCT_0 \) networks. Each of them is comprised of at most 3\( P \)\( (3P-1)/2 \) half-lattice two-poles which can be realized almost by inspection [2].

This completes the proof of item (c) and Theorem 2 altogether.

Now, we shall consider and formulate the steps of algorithm for realization of \( RRF \) matrix \( Y(s)=[N_{ij}]_{P \times P} \) by an active, \( BRCT_0 \)-P-port network.

- Firstly, we must select symmetric, \( P \)-th order \( RRF \) matrix \( Y(s)=[x_{ij}]_{P \times P} \) with strictly dominant MCSFE. \( Y(s) \) must be realizable by \( BRCT_0 \)-P-port network. Therefore, we have \( x_{ij} \geq x_{ij}^0 + \varepsilon \) (i=1, 2, ..., \( P \); i \( \neq j \)), where \( \varepsilon > 0 \), depending on whether the point at infinity is pole or common-point of \( Y_{11}(s) \), respectively [2, 6-8, 10].
- In addition, \( Y_{11}(s) \) must assure the factorizability of \( Y_{11}(s) \)-\( Y(s)=[D_{ij} \cdot q \cdot N_{ij}]_{P \times P} \) — according to Theorem 1.
- We shall deliberately suppose that \( x_{ij}^0 \) is independent of \( j (j=1, 2, \ldots , P) \). Then we have \( L = \max \{D_{ij}^q + \max (N_{ij}^q)\}_{i=1,2, \ldots , P} \), \( q \neq 0 \), provided that \( L \leq \max \{D_{ij}^q, \max (N_{ij}^q)\}_{i=1,2, \ldots , P} \).
- Strict dominancy of MCSFE in \( Y_{11}(s) \) with sufficiently large dominancy-margin [2, 9, 10] can always be achieved by multiplication of its diagonal entries with sufficiently large, real, positive constant \( \rho \).

\[ Y_{11}(s) = \frac{1}{q} \left[ \begin{array}{cccc} \rho \cdot x_{11} & \cdots & \cdots & x_{1P} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ x_{P1} & \cdots & \cdots & \rho \cdot x_{PP} \end{array} \right], \quad (40) \]

so that \( \det (P(s)) \) can be represented as follows,

\[ \det (P) = \det \left[ D_{ij} + q \cdot N_{ij} \right] = \rho \left[ D_{ij}^q + \frac{R}{\rho^p} \right], \quad (41) \]

\[ R(s, \rho) \leq P \cdot x_{ij}^0 < P \cdot L \quad \lim_{\rho \to \infty} \frac{R(s, \rho)}{\rho^p} = 0. \]

When \( \rho \to \infty \), \( P \cdot x_{ij}^0 \) zeros of \( P(s) \) approaches the zeros of product \( x_{ij}^0 \cdot x_{ij}^0 \cdot \cdots \cdot x_{PP} \), then all of them should be selected to be real, simple and distinct from zeros of both polynomials \( D(s) \) and \( q(s) \). Thuswith, det \( P(s) \) can be produced to have at least \( P \cdot x_{ij}^0 \) real, distinct zeros. Then, for extraction of \( k \) linear matrix factors from \( P(s) \) it is sufficient after extraction of the \( (k-1) \)-th factor to be:

\[ P \cdot x_{ij}^0 - P \cdot (k-1) > (P-1) \cdot [x_{ij}^0 + L_q -(k-1)]. \quad (42) \]

This follows straightforwardly from (19). If we assume now \( k=L_q \), the inequality (42) reduces to \( x_{ij}^0 > P \cdot L_q-1 \) and can be satisfied by selecting \( x_{ij}^0 = P \cdot L_q \). It is shown thuswith that if det \( P(s) \) has at least \( P \cdot x_{ij}^0 \) real, distinct zeros, the regular \( \lambda \)-matrix \( P(s) \) can be represented as product of regular \( \lambda \)-matrices \( P(s) \) and \( P(s) \), having degrees \( p_{ij} = x_{ij}^0 \cdot L_q \) and \( p_{ij} = L_q \), respectively. We must recall that \( x_{ij}^0 > P \cdot L_q-1 \) is only a sufficient condition (and not necessary one) for factorization of \( P(s) \).

- Since \( Y_{11}(s)=[x_{ij}]_{P \times P} \) must have strictly dominant MCSFE, it follows that is also necessary to be \( x_{ij}(0) \neq 0 \) (i=1, 2, ..., \( P \)). Otherwise, the strict dominancy of MCSFE of \( Y_{11}(s) \) may be violated.

Now we formulate steps of realization algorithm:

(a) Select \( Y_{11}(s)=[x_{ij}]_{P \times P} \) with strictly dominant MCSFE. \( Y_{11}(s) \) must be realizable by a \( BRCT_0 \)-network. Zeros of \( x_{ij}(s) \) (i=1, 2, ..., \( P \)) should be selected real, simple and distinct from zeros of polynomials \( D(s) \) and \( q(s) \). All polynomials \( x_{ij}(s) \) may have the same degree and in addition, \( x_{ij}(0) \neq 0 \) (i=1, 2, ..., \( P \)). These diagonal entries possibly should be multiplied by sufficiently great real \( \rho > 0 \) in order to assure that the \( \lambda \)-matrix \( P(s)=[D_{ij} \cdot q \cdot N_{ij}]_{P \times P} \) is regular and to have at least \( P \cdot x_{ij}^0 \) real, simple and distinct zeros.

(b) Let \( L_q = \max \{D^q, \max (N_{ij}^q)\}_{i=1,2, \ldots , P} \). Then, realize the factorization \( P(s)=P_i(s) \cdot P_j(s) \), where degrees
of regular \(\lambda\)-matrices \(P_1(s)\) and \(P_2(s)\) are \(p_1=x_1^0=P_1L_1\) and \(p_2L_2\), respectively.

(c) Select the real-valued voltage-amplifier gains \(A_i\) as \(|A_i| > 1 (i=1,2,\ldots,P)\). Then apply the identification procedure, either of the Case 1\(^6\), or of the Case 2\(^6\) to produce necessary block-matrices of \(Y(s)\), which is realizable by passive \(BRCT_0\) network. Realization of \(Y^*\) is simple and straightforward \([2]\). And finally, by interconnection of \(VCVS\) and previously produced \(BRCT_0\) 3P-port network \(N_P\) the balanced, active, \(P\)-port, transformerless, RC network which realizes the admittance matrix \(Y(s)\) is finally constructed.

5. A NUMERICAL EXAMPLE

By using the procedure proposed let us realize 2 \(\times\) 2 admittance matrix:

\[
Y(s) = \begin{bmatrix} N_{12} \\ D(s) \end{bmatrix} = \begin{bmatrix} 2s+1 & s \\ s & 2s-1 \end{bmatrix}, \quad D(s) = s + 1. \tag{43}
\]

Although \(Y(s)\) is symmetric, it can not be realized by passive \(RLCT\) network, since the entry \(y_{22}(s)=N_{22}(s)/D(s) = (2s-1)/(s+1)\) is not the sum of moduli of entries in the same row (column), i.e. \(z_{0} \geq |z_{1}|+|z_{2}|+\ldots+|z_{3}|\) or \(|z_{0}|-|z_{1}|-\ldots-|z_{3}|\) \([2]\). Since each row of matrix (48) must be dominant, therefore we obtain that \(c \geq 6\). According to (30) and by algorithm we have:

\[
Y_{22} + Y_{23} \cdot A = \frac{1}{c^2} \cdot U_2, \quad D_0 = d_0 \cdot U_2, \quad D_1 = d_1 \cdot U_2 , \tag{49}
\]

where \(d_0, d_1 > 0\). Then, \(Y_{22}\) and \(Y_{23}\) can be identified as:

\[
Y_{22}(s) = \begin{bmatrix} d_0 + \frac{1}{c^2} & 0 \\ 0 & d_0 + \frac{1}{c^2} \end{bmatrix} + \begin{bmatrix} d_0 & 0 \\ 0 & d_0 \end{bmatrix} \frac{s}{s+1}, \tag{50}
\]

\[
Y_{23}(s) = \begin{bmatrix} \frac{d_0}{2} & 0 \\ 0 & \frac{d_0}{2} \end{bmatrix} + \begin{bmatrix} \frac{d_0}{2} & 0 \\ 0 & \frac{d_0}{2} \end{bmatrix} \frac{s}{s+1}. \tag{51}
\]

The third and forth row of coefficient-matrices in \(Y^*\) are dominant if the same holds for submatrix \([Y_{12} | Y_{21} | Y_{22} | Y_{23}]\). One can easily show that the desired dominancy can be realized through selection \(c_{10} \geq 6, d_0=2\) and \(d_1=1\). Dominancy of the fifth and sixth row of \(Y^*(s)\) can be readily achieved by selecting \(Y_{13}\) as diagonal matrix, as for example:

\[
Y_{13}(s) = \begin{bmatrix} 13/12 & 0 \\ 0 & 7/6 \end{bmatrix} + \begin{bmatrix} 3/4 & 0 \\ 0 & 2/3 \end{bmatrix} \frac{s}{s+1}. \tag{52}
\]

Herewith, we completely accomplished construction of the sought matrix \(Y^*\), which can be realized straightforwardly (almost by inspection), by a passive, balanced, transformerless, RC, six-port network:
Realization details of dominant matrices are discussed elsewhere [2]. Network realizations of this class of matrices are established exclusively from half-lattices with only serial, or crossed arms (pair of branches), inserted between each pair of nodes (i.e., network ports). By use of (52) we find, for example, that the network fragment between ports 2 and 3 looks like as is depicted in Fig 3. Thereon are denoted the dimensionless values of conductances \((G_n)\) and capacitances \((C_n)\), normalized with respect to assumed values of angular frequency \((\omega_0)\) and the impedance level \((R_0)\). Physical parameter values of these elements are obtained by denormalization process: the values of conductances are calculated as \(G = G_n/R_0\), whereas the values of capacitances are obtained as \(C = C_n/(R_0 \omega_0)\).

Figure 3. The fragment of active BRCT\(_5\) network which realizes the admittance matrix (43)

6. CONCLUSIONS

In paper is presented a new proof of fundamental theorem in network synthesis of admittance matrices of real, rational functions in complex frequency by active, multiport, transformerless, balanced RC networks. It has been shown that the necessary condition for network existence is also a sufficient one if active elements used are balanced voltage-amplifiers. A new procedure for realization of RRF admittance matrices is given also by use of minimum number of balanced VCVS. The procedure proposed relies on a new theorem on representation of regular, polynomial matrix as product of two polynomial matrices with selected degrees. The obtained results are the most general in nature and can easily be applied in transformerless, multiport, RC active network synthesis.

REFERENCES