Thin walled I-beam under complex loads - Optimization according to stress constraint

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Optimization of a thin-walled open section I-beam loaded in a complex way, subjected to the bending and to the constrained torsion, is considered. From the general case when bending moments about both principal axes appear simultaneously with the bimoment, some particular cases can be considered depending on the loading case. The problem is reduced to the determination of minimum mass i.e. minimum cross sectional area of structural thin-walled beam elements of proposed shape, for given complex loads, material and geometrical characteristics. That is why the area of the cross section is taken as the objective function. The ratios of thickness and length of the parts of the cross section are assumed to be non constant. The stress constraint is introduced. The starting points during the formulation of the basic mathematical model are the assumptions of the thin-walled beam theory from one side and the basic assumptions of the optimum design from the other. The Lagrange multiplier method is used. Solutions of analitically obtained expressions for the mathematical model, numerical solutions, as well as the saved mass, are calculated for three loading cases.

Keywords: *optimization, thin-walled beam, optimal dimensions, saved mass, stress constraint.*

1. INTRODUCTION

Optimization is a mathematical process through which the set of conditions is obtained giving as the result the maximum or minimum value of a specified function. In the ideal case, one would like to obtain the perfect solution for the considered design situation. But in the reality, one can only achieve the best solution.

The quantities numerically calculated during the process of obtaining the optimal solution are called the design variables.

The process of selecting the best solution from various possible solutions must be based on a prescribed criterion called the objective function. It is represented by a mathematical equation that embodies the design variables to be minimized or maximized.

The total region defined by the design variables included in the objective function is called the design space and it is limited by the constraints.

Many studies have been made on the optimization problems treating the cases where geometric configurations of structures are specified and only the dimensions of members, such as areas of members cross sections, are determined in order to attain the minimum structural weight or cost. Many methods have been developed for the determination of the local minimum point for the optimization problem [2, 5, 8, 9, 10].

One of very often used thin walled profiles in steel structures, I-cross section, is considered in this paper as the object of the optimization.

The determination of optimal dimensions is a very important process but not always the simple one.

The aim of this paper is the determination of the minimum mass of the beam.

2. DEFINITION OF THE PROBLEM

It is assumed that the load can be applied to the I-beam in an arbitrary way.

The cross section of the beam (Fig. 1) is supposed to have flanges of mutually equal widths $b_1=b_3$, and thicknesses $t_1=t_3$, and the web of width b_2 and thickness t_2. The ratios of thickness and widths of flanges are treated as not constant quantities.

The load is applied in two longitudinal planes, which are parallel to principal axes X_i (i=1,2) of the cross section. If applied in such a way the loads will produce the bending moments acting in above mentioned two planes parallel to the longitudinal axis of the beam and as the consequence of such a kind of loads the effects of the constrained torsion will appear in the form of the bimoments producing the corresponding stresses [3, 6].

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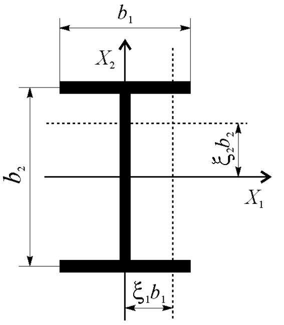


Figure 1. Cross section.

The determination of the minimum mass of the beam reduces in another way to the determination of the minimum cross-sectional area described by (1)

$$A = A_{\min} \ , \tag{1}$$

for the given loads and material and geometrical properties of the considered beam and its section.

3. OBJECTIVE FUNCTION

If the coefficients (2)

$$\mu_i = \frac{t_i}{b_i} \neq \text{const.}\ , \quad (i = 1,2) \tag{2}$$

are introduced it can be seen from Fig. 1 that the area of the cross section is given by (3)

$$A = \sum b_i t_i = \sum \mu_i b_i^{\,2}, \quad (i = 1,2) \tag{3}$$

and it will be treated as an objective function in the considered problem.

4. CONSTRAINTS

The normal stresses will be taken into account in the considerations that follow and that is why the constraints treated in the paper are the stress constraints.

The normal stresses are the consequences of the bending moments M_{X_i} (i=1,2), and of the bimoment B that appears if the constrained torsion exists and they will be denoted by σ_{X_i} (i=1,2), and σ_ω respectively [3, 6].

If the allowable stress is denoted by σ_0 the constraint function can be written as

$$\varphi_1 = \varphi(\sigma) = \sigma_{X_1 \max} + \sigma_{X_2 \max} + \sigma_{\omega \max} \leq \sigma_0 \,. \tag{4}$$

The maximal normal stresses, are defined [3, 6] in the form

$$\sigma_{X_i \max} = M_{X_i} / W_{X_i} \ , \quad (i = 1,2) \tag{5}$$

$$\sigma_{\omega \max} = B / W_\omega \,, \tag{6}$$

where W_{X_i} (i=1,2) are the section moduli and W_ω is the sectorial section modulus for the considered cross section.

After the introduction of (5) and (6) into the expression (4), the constraint function becomes (7)

$$\varphi = \frac{M_{X_1}}{W_{X_1}} + \frac{M_{X_2}}{W_{X_2}} + \frac{B}{W_\omega} - \sigma_0 \leq 0 \,. \tag{7}$$

If the ratio

$$z = b_2 / b_1 \ , \tag{8}$$

is the optimal relation of the parts of the considered cross section and if

$$\psi = t_2 / t_1 \ , \tag{9}$$

the constraint function (7) will be reduced in the considered case to the form

$$\begin{aligned} \varphi = 6M_{X_1} \frac{1}{t_1 b_1 b_2 (6 + \psi z)} + 3M_{X_2} \frac{1}{t_1 b_1^2} + \\ + 6\mathrm{B} \frac{1}{t_1 b_1^2 b_2} - \sigma_0 \leq 0 \,. \end{aligned} \tag{10}$$

After the diferentiation of the expression (10) with respect to the variables b_1 и b_2, the following expressions are obtained

$$\begin{aligned} \frac{\partial \varphi}{\partial b_1} = -36 M_{X_1} \frac{1}{t_1 b_1^2 b_2 (6 + \psi z)^2} - \\ - 6 M_{X_2} \frac{1}{t_1 b_1^3} - 12 B \frac{1}{t_1 b_1^3 b_2} \quad , \end{aligned} \tag{11}$$

$$\frac{\partial \varphi}{\partial b_2} = -12 M_{X_1} \frac{1}{t_1 b_1 b_2^2} \frac{(3 + \psi z)}{(6 + \psi z)^2} - 6B \frac{1}{t_1 b_1^2 b_2^2} \,. \tag{12}$$

5. LAGRANGE MULTIPLIER METHOD

Applying the Lagrange multiplier method [2, 5, 8, 9, 10, 11] to the vector which depends on two parameters b_i (i=1,2) the system of equations

$$\frac{\partial}{\partial b_i}(A + \lambda \varphi) = 0, \quad (i = 1,2) \tag{13}$$

will be obtained and, after the elimination of the multiplier λ from (13), it obtains the form (14)

$$\frac{\partial A}{\partial b_i} \frac{\partial \varphi}{\partial b_j} = \frac{\partial A}{\partial b_j} \frac{\partial \varphi}{\partial b_i}, \quad (i \neq j, \ i = 1, j = 2). \tag{14}$$

6. BIMOMENT EXPRESSED THROUGH THE BENDING MOMENTS

If the bending moments are acting in planes parallel to the longitudinal axis the bimoment as their consequence will appear and it can be expressed as the product (15) of the bending moments and the eccentrities of their planes ξ_i b_i (i=1,2) measured from the principal axes (Fig. 1) in the following way [3, 6]

$$B = \sum \xi_i b_i M_{X_i} \ , \quad (i = 1,2) \tag{15}$$

In the considered case when the I-beam is the object of the optimization, the equation (14) will definitelly be reduced to the equation

$$\sum_{k=0}^{k=4} c_k z^k = 0 \,. \tag{16}$$

The coefficients c_k in (16) are defined by (17)

$$
\begin{aligned}
c_0 &= -12(1+6\xi_1), \\
c_1 &= 2\left[\psi(1+24\xi_1)-36\xi_2\frac{M_{X_2}}{M_{X_1}}\right], \\
c_2 &= 2\psi\left[11\psi\xi_1+6(3+4\xi_2)\frac{M_{X_2}}{M_{X_1}}\right], \\
c_3 &= 2\psi^2\left[\psi\xi_1+(6+11\xi_2)\frac{M_{X_2}}{M_{X_1}}\right], \\
c_4 &= \psi^3(1+2\xi_2)\frac{M_{X_2}}{M_{X_1}}.
\end{aligned} \tag{17}
$$

7. THE LOADING CASES

From the general case when bending moments about both principal axes appear simultaneously with the bimoment as their consequence, some particular cases will be considered depending on the loading case.

As an example in this chapter an I-beam fixed at one end will be considered and it will be subjected to two loads: concentrated bending moment and concentrated force acting at the free end of the beam. The results obtained through the analytical approach are given here, and later in chapter 9, these cases are treated numerically using the Finite Element Method (FEM) [1].

7.1. Beam loaded by a concentrated bending moment at its free end

The concentrated bending moment will be introduced in two different ways (Loading cases 1 and 2) presented in Figs. 2 and 3. Three values ψ = 0.5; 0.75; 1 for the relation (9) are assumed. The eccentricities of the moment planes from (15) are assumed as ξ_1, ξ_2 = 0, 0.2, 0.4, 0.6, 0.8, 1.0, 3.0, 5.0, or in another way $0\le\xi_1\le5$; $0\le\xi_2\le.5$.

Loading case 1

$M_{X_2} = 0$

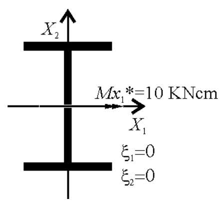


Figure 2. Bending moment M_{X_1} in the plane ξ_1=0

Loading case 2

$M_{X_2} = 0$

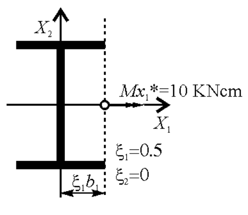


Figure 3. Bending moment M_{X_1} in the plane ξ_1=0.5

The optimal ratios $z=b_2/b_1$ defined by (8) obtained from the equation (16) and the relations between z and the eccentricities ξ_1 and ξ_2 for Mx_2/Mx_1=0 and t_2/t_1=0.5 are shown in Figs. 4 and 5. The optimal ratios z for t_2/t_1=0.75 and 1.0 are given in Tables 1 and 2. The columns in Tables 1 and 2 are given in shortened form because the ratios z have same values for each ξ_2.

Table 1. Optimal *z* for $M_{X_2}/M_{X_1}=0,\ t_2/t_1=0.75$

$\downarrow\xi_2$ \ $\xi_1 \rightarrow$	0	0.2	0.4	0.6	0.8	1	3	5
0	8	1.89	1.64	1.54	1.49	1.46	1.38	1.36
⋮	…							
5	8	1.89	1.64	1.54	1.49	1.46	1.38	1.36

Table 2. Optimal *z* for $M_{X_2}/M_{X_1}=0,\ t_2/t_1=1$

$\downarrow\xi_2$ \ $\xi_1 \rightarrow$	0	0.2	0.4	0.6	0.8	1	3	5
0	6	1.42	1.23	1.16	1.12	1.09	1.03	1.02
⋮	…							
5	6	1.42	1.23	1.16	1.12	1.09	1.03	1.02

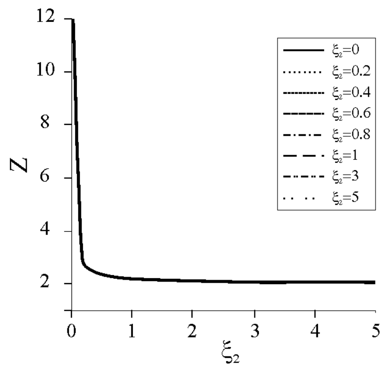


Figure 4. Relation between *z* and ξ_1

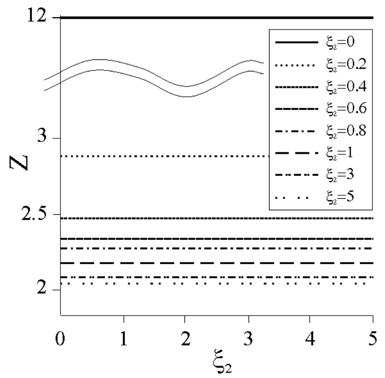


Figure 5. Relation between *z* and ξ_2

From Figs. 4 and 5 and from the results presented in Tables 1 and 2, it is obvious that the quantity *z* decreases with the increase of the eccentricity ξ_1 and that it does not depend on the eccentricity ξ_2.

7.2. Beam loaded by a concentrated force at its free end

For the previously defined models when the beam was loaded by only one concentrated bending moment, the maximal normal stress values at the fixed end of the considered cantilever I-beam of the length L = 150 cm loaded by the concentrated force F* passing through the shear center plane (Fig. 6) are presented (Loading case 3). In the case of an I-beam the shear center plane coincides with the web.

Loading case 3

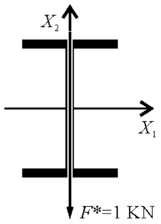


Figure 6. Concentrated force along the web

The results for the ratios (8) $z = b_2/b_1$ obtained from the equation (16) are the same as the results for the loading case 1 and they are also presented in Figs. 4 and 5 and in Tables 1 and 2.

8. DETERMINATION OF THE MINIMUM CROSS SECTIONAL AREA

The cross sectional geometrical characteristics are calculated taking into account the initial dimensions of the I-section: b_1=5.5 cm, b_2=9.2 cm, t_1=0.8 cm, t_2=0.8 cm. The length of the considered cantilever I-beam is L=150 cm.

For the given loading cases (Figs. 2, 3 and 6) and for the defined geometry of the profile, the initial stresses are calculated.

The problem is considered in two ways:

a) The optimal dimensions of the cross section $b_{1optimal}$ and $b_{2optimal}$ are obtained by equalizing initial and optimal areas ($A_{initial}$=$A_{optimal}$) and by using the calculated optimal relation z from the expressions derived in this paper. In that case the normal stress lower than the initial one is obtained ($\sigma_{optimal}$<$\sigma_{initial}$) and it represents the model used for the control (Table 3).
b) The optimal values b_1^* and b_2^* are obtained for the given loading cases using the calculated optimal ratio $z = (b_2/b_1)_{optimal}$. From the condition prescribing that the stresses must be lower than the allowable i.e. initial stress, the optimal values are obtained by comparing the stress defined by the optimal geometrical characteristics to the initial stress (optimal model).

Starting from the optimal cross sectional dimensions (b_1^* and b_2^*), the optimal (minimum) cross sectional area A_{min} is calculated for the given loading case and the results including the saved material are given in Table 3.

Table 3. Normal stresses and saved mass

Loading case	z initial	z optimal
1	1.6727	7.3846
2		1.4602
3		7.3846

Loading case	$W_{x\ init.}$ [cm^3]	$W_{\omega\ init.}$ [cm^4]	σ init. [kN/cm^2]
1	49.649		0.2014
2		37.107	0.9427
3			3.02

Loading case	$W_{x\ contr.}$ [cm^3]	$W_{\omega\ contr.}$ [cm^4]	σ contr. [kN/cm^2]
1	62.988		0.15876
2	47.033	37.947	0.9373
3	62.988		2.38

Loading case	$W_{x\ opt.}$ [cm^3]	$W_{\omega\ opt.}$ [cm^4]	σ opt. [kN/cm^2]
1	49.654		0.2014
2	46.838	37.711	0.9427
3	49.654		3.02

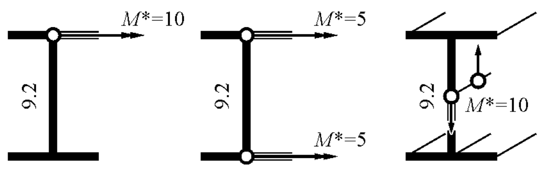
Loading case	A init. [cm^2]	A opt. [cm^2]	Saved mass [%]
1	14.26	12.6	11.64
2		14.23	0.217
3		12.6	11.64

9. APPLICATION OF THE FINITE ELEMENT METHOD

Loading cases 1, 2 and 3 presented in the previous chapters now are treated also by the Finite Element Method (FEM) using the programme KOMIPS [4]. The cross sectional geometrical characteristics are calculated taking into account the initial dimensions of the I-section considered in the Chapter 8.

The model of the beam having the length L=150 cm consists of 360 2-D plate finite elements. The flanges are divided into 90 elements each (2 elements through the width and 45 elements along the beam), and the web is divided into 180 elements (4 elements through the width and 45 elements along the beam). The elements are numerated starting from the fixed end towards the free end of the beam. The first 90 elements are in the upper flange, next 90 elements are in the lower flange and last 180 elements are in the web.

When the FEM is applied the introduction of the concentrated bending moment in the loading case 1 is modeled in three ways: loading cases 4, 5 and 6 presented in Fig. 7*a*, 7*b* and 7*c*.



(a) Loading case 4 *(b)* Loading case 5 *(c)* Loading case 6

Figure 7. Concentrated bending moment M_{X_1} in the plane ξ_1=0

Loading case 4: Concentrated bending moment M^*=10 kNcm is introduced in the nodal point situated at the connection of the upper flange and the web (Fig. 7*a*).

Calculated normal stresses [kN/cm^2] in first four element layers of the upper flange (elements no. 83 ÷ 90) are given in Table 4.

Table 4. Loading case 4 - Normal stresses

89	8.43	**87**	3.12	**85**	0.29	**83**	0.21
90	8.43	**88**	3.12	**86**	0.29	**84**	0.21

In the case *a)* the maximal stress concentration appears at the place of the introduction of the load, in the elements 89 and 90. In the fourth line of elements the stresses corespond to analitically obtained values at the distance of 1.45* b_2 from the place of the introduction of the loads.

Loading case 5: Two concentrated bending moments M*=5 kNcm each, having total value M*=10 kNcm, are introduced in the nodal points situated at the connections of the horizontal flanges and the web (Fig. 7*b*).

Calculated normal stresses [kN/cm^2] in the same elements no. 83 ÷ 90 are given in Table 5.

Table 5. Loading case 5 - Normal stresses

89	4.24	**87**	1.63	**85**	0.24	**83**	0.21
90	4.24	**88**	1.63	**86**	0.24	**84**	0.21

The same results are obtained for the elements in the lower flange.

In the case *b*) the maximal stress concentration appears in the elements 89, 90, 179 and 180, but it is 50% lower than in the case *a*).

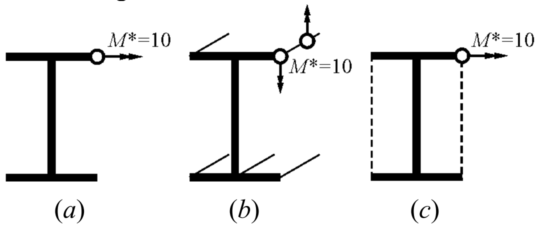
In the fourth line of elements the stresses corespond to analitically obtained values again at the distance of 1.45* b_2 from the place of the introduction of the loads.

Loading case 6: The concentrated bending moment M*=10 kNcm is represented by the couple produced by two paralel vertical concentrated forces F*=3kN introduced in the nodal points situated in the centroid and on the centroidal axis at the distance of 3,33 cm from the end of the beam (Fig. 7*c*).

The maximal normal stress σ=1.11 kN/cm^2 appears in the element no. 358 and in all other elements the stresses are approximately $\sigma \approx 0.2$ kN/cm^2.

In the case *c*) the stress concentration is minimal if compared to the cases *a*) and *b*) and the highest value appears in the element 358 – at the place of the introduction of the load.

When the FEM is applied, the introduction of the concentrated bending moment in the loading case 2 is modeled in three ways: loading cases 7, 8 and 9 presented in Fig. 8*a*, 8*b* and 8*c*.



Loading case 7 Loading case 8 Loading case 9

Figure 8. Concentrated bending moment M_{X_1} in the plane ξ_1=0.5

Loading case 7: Concentrated bending moment M*=10 kNcm is introduced in the model in the nodal point situated at the end of the upper flange (Fig. 8*a*).

Calculated normal stresses [kN/cm^2] in first three element layers of the upper flange (elements no. 85 ÷ 90) are given in Table 6.

Table 6. Loading case 7 - Normal stresses

89	4.63	**87**	4.09	**85**	0.89
90	15.41	**88**	2.37	**86**	1.08

In the case *a)* the maximal stress concentration appears at the place of the introduction of the load, in the element 90. In the third line of elements the stresses corespond to analitically obtained values at the distance of 1.08* b_2 from the place of the introduction of the loads.

Loading case 8: The concentrated bending moment M*=10 kNcm is represented by the couple produced by two paralel vertical concentrated forces F*=3 kN introduced in the nodal points situated at the end of the upper flange and at the distance of 3,33 cm from the end of the beam (Fig. 8*b*).

Calculated normal stresses [kN/cm^2] in same elements no. 85 ÷ 90 are given in Table 7.

Table 7. Loading case 8 - Normal stresses

89	2.76	**87**	2.22	**85**	1.02
90	16.18	**88**	3.34	**86**	1.09

In the third line of elements the stresses corespond to analitically obtained values at the distance of 1.08* b_2 from the place of the introduction of the loads.

Loading case 9: The concentrated bending moment M*=10 kNcm is introduced in the same way as in the case *a*), but the end of the cantilever beam is stiffened by the vertical rectangular plate (Fig. 8*c*).

Calculated normal stresses [kN/cm^2] in the elements no. 85 ÷ 90 are given in Table 8.

Table 8. Loading case 9 - Normal stresses

89	2.14	**87**	1.92	**85**	0.58
90	7.91	**88**	2.67	**86**	1.11

In the third line of elements the stresses corespond to analitically obtained values again at the distance of 1.08* b_2 from the place of the introduction of the loads.

When the FEM is applied, the introduction of the concentrated force along the web in the loading case 3 is modeled in only one way - loading case 10 presented in Fig. 9.

Loading case 10: Two concentrated vertical forces F*=0.5 kN, each having total value F*=1kN, are introduced in the model in the nodal points situated on the centroidal axis on both sides of the web (Fig. 9). Only in this case the I-beam is modeled using 3-D finite elements.

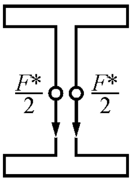


Figure 9. Concentrated forces along the web.

The results obtained by program KOMIPS corespond to analitically obtained values (Table 3).

The results are presented in Table 9 for the initial, the control and the optimal model.

Table 9. Results obtained by FEM corespond to analitically obtained values

Model	Results -KOMIPS	Results - Table 3
Initial	$\sigma = 2.96$ kN/cm^2	$\sigma = 3.02$ kN/cm^2
Control	$\sigma = 2.28$ kN/cm^2	$\sigma = 2.38$ kN/cm^2
Optimal	$\sigma = 2.82$ kN/cm^2	$\sigma = 3.02$ kN/cm^2

10. CONCLUSION

On the basis of the proposed optimization procedure it is possible to calculate the optimal ratios between the parts of the considered thin walled profiles in a very simple way.

For all loading cases it is possible to find the decreased level of the stresses in the Control model as well as the saved mass of material with respect to the initial stress limits.

The maximal normal stresses depend on the way of the introduction of the loads (stress concentration appears around the place of introduction of the loads).

The results obtained by the Finite Element Method show and prove the existence of Saint-Venant principle. As it is known the influence of the stress concentration disappears at the distance between one and two cross sectional dimensions.

It is also possible to calculate the saved mass of the used material for different loading cases.

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СЛОЖЕНО ОПТЕРЕЋЕНИ ТАНКОЗИДИ НОСАЧ I-ПРОФИЛА - ОПТИМИЗАЦИЈА ПРИ НАПОНСКОМ ОГРАНИЧЕЊУ

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Разматрана је оптимизација сложено оптерећених танкозидих носача попречних пресека облика I-профила изложених савијању и ограниченој торзији. Из општег случаја, када моменти савијања делују око обе главне тежишне осе истовремено са бимоментом, издвојени су неки посебни случајеви који се разматрају у зависности од случаја оптерећења. Проблем је редукован на одређивање минималне масе, т.ј. минималне површине предложеног облика попречног пресека танкозидог носача, за дата сложена оптерећења, материјал и геометријске карактеристике. Због тога је површина попречног пресека изабрана за функцију циља. Претпоставља се да однос дебљине и ширине појединих делова попречног пресека није константан. Уведено је напонско ограничење. При формирању основног математичког модела пошло се од претпоставки теорије танкозидих штапова са једне стране и основних претпоставки проблема оптималног пројектовања са друге. Коришћена је метода Лагранжовог множитеља. Резултати аналитички добијених једначина за математички модел, нумеричка решења, као и уштеда масе, израчунати су за три случаја оптерећења.