Buckling of Stepped Thickness Plates in the Theory of Plasticity

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The problem of buckling has been treated for a long time as a single rectangular field with different boundary conditions loaded with inplate loads and usualy applying Levy method, when two opposite edges were always simply supported. This paper is concerned with the plastic buckling of rectangular plates in the region of plasticity. In this paper, it has been treated the elasto-plastic deformations for two coupled plates with different thicknesses, loaded with inplane constant forces N_x = const.

Keywords: *buckling, plates, plasticity, deformations.*

1. INTRODUCTION

The problem of buckling has been treated for many years as one rectangular field with different boundary conditions loaded with in-plate loads N_x, and N_y and usually applying Levy method, when two opposite edges were always simply supported.

In plastic region A. A. Ilyushin [1] and E. Z. Stowell [2], using their differential equations solved many buckling problems in elasto-plastic region, but always with plates of constant thickness. Now, in this paper, the author has solved the problem of buckling when two rectangular plates form the unity, but of different thicknesses. All sides are simply supported and pressed along two parallel opposite sides with N_x = const.

It is known, Fig. 1, that in the plastic region the angle α_0 corresponds to modulus of elasticity (E), elastic region, and direction ON$_0$, determined by angle α_0 gives "cutting modulus" of elasticity (E_c^0), where the point N$_0$ gives the initial appearance of plasticity [3]. The value of angle α_k gives "tangential modulus of plasticity".

$$\mathrm{E}_k^0 = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} . \tag{1}$$

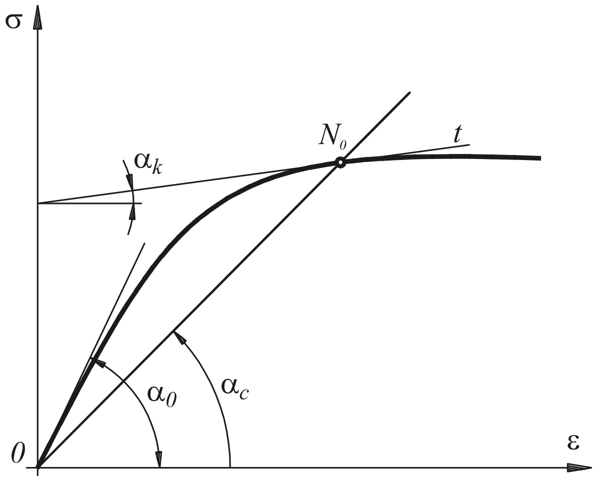

Figure 1. Relation between (α, α_c, α_k) and (E, E_c, E_k)

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The direction ON determines the angle α_c which determines primary modulus of elasticity

$$E_c^0 = \frac{\sigma}{\varepsilon} . \tag{2}$$

According to the hypothesis that the radius of the Mises circle may be written, as it is known, as generalized stress, effective stress or equivalent stress.

In the case of complete stresses the intensity of stress σ_i and intensity of deformation ε_i are introduced in the forms

$$\sigma_i = \frac{1}{\sqrt{2}} \cdot \sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)} \tag{3}$$

and

$$\varepsilon_i = \frac{\sqrt{2}}{3} \cdot \sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \left(\varepsilon_y - \varepsilon_z\right)^2 + \left(\varepsilon_z - \varepsilon_x\right)^2 + \frac{3}{2}\left(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2\right)} . \tag{4}$$

2. FORMULATION OF THE PROBLEM

In our case we are solving the problem of stability of plates, when we have small elasto-plastic deformations, using partial differential equations of A. A. Ilyushin [1,2]

$$\left(1 - \frac{3}{4}\frac{1-s}{1-r}\frac{\sigma_x^2}{\sigma_i^2}\right)\frac{\partial^4 w}{\partial x^4} + 2\left(1 - \frac{3}{4}\frac{1-s}{1-r}\frac{\sigma_x\sigma_y + 2\tau^2}{\sigma_i^2}\right) \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + \left(1 - \frac{3}{4}\frac{1-s}{1-r}\frac{\sigma_y^2}{\sigma_i^2}\right)\frac{\partial^4 w}{\partial y^4} - 3\frac{1-s}{1-r}\frac{\tau}{\sigma_i^2} \cdot \left(\sigma_x \frac{\partial^4 w}{\partial x^3 \partial y} + \sigma_y \frac{\partial^4 w}{\partial x \partial y^3}\right) + \frac{1}{(1-r)D'}\Pi(\sigma, w) = 0 . \tag{5}$$

In (5) is introduced the sign

$$\Pi(\sigma, w) = \sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2} . \tag{6}$$

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In our case we consider two plates (1) and (2) (Fig. 2), it is plate of dimensions $a \times b$ which has contact along the edge $y = \eta$, $(0 < \eta < b)$.

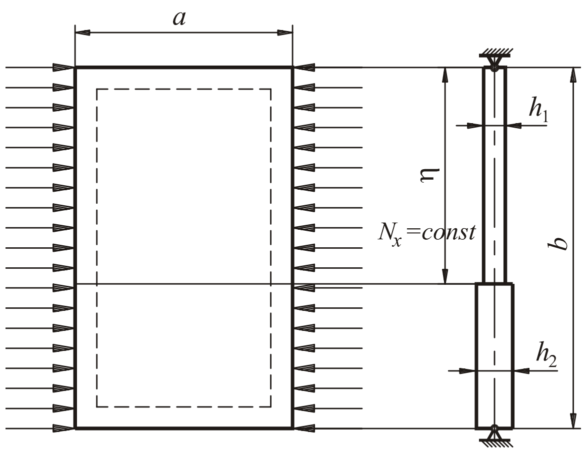

Figure 2. Two coupled plates with load N_x = const.

Now, we have

- plate (1): $0 \le x \le a$; $0 \le y \le \eta$
- plate (2): $0 \le x \le a$; $\eta \le y \le b$.

In (5) are given: $r = 1 - \varphi_c$, $s = 1 - (\varphi_c - \varphi_k)$.

Along the edges $y = 0$ and $y = b$ we could have any kind of boundary conditions. Along the plates (1) and (2), for $x = 0$ and $x = a$ are pressed with forces N_x = const. and they have thicknesses h_1 and h_2. The other forces are $N_y = 0$ and $N_{xy} = 0$.

In actual case is used modified differential equation given by E. Z. Stowell

$$\left[1-\frac{3}{4}\left(1-\frac{\varphi_k}{\varphi_c}\right)\frac{\sigma_x^2}{\sigma_i^2}\right]\frac{\partial^4 w}{\partial x^4}+2\left[1-\frac{3}{4}\left(1-\frac{\varphi_k}{\varphi_c}\right)\frac{\sigma_x\sigma_y+2\tau^2}{\sigma_i^2}\right]\cdot$$

$$\cdot\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\left[1-\frac{3}{4}\left(1-\frac{\varphi_k}{\varphi_c}\right)\frac{\sigma_y^2}{\sigma_i^2}\right]\frac{\partial^4 w}{\partial y^4}-3\left(1-\frac{\varphi_k}{\varphi_c}\right)\cdot$$

$$\cdot\left(\sigma_x\frac{\partial^4 w}{\partial x^3\partial y}+\sigma_y\frac{\partial^4 w}{\partial x\partial y^3}\right)+\frac{h}{D_c'}\Pi(\sigma,w)=0 \qquad (7)$$

where values for aluminium and for steel are given in [5].

For our case, given with (Fig. 2), (7) is reduced to:

$$\left(\frac{1}{4}+\frac{3}{4}\frac{\varphi_k}{\varphi_c}\right)\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\frac{\sigma_x h}{D_c'}\frac{\partial^2 w}{\partial x^2}=0 \quad (8)$$

where $\varphi_c = \dfrac{E_c}{E}$, $\varphi_k = \dfrac{E_k}{E}$, $D_c' = D_c'^1 = D_1$ or $D_c' = D_c'^2 = D_2$.

According to Morris-Lavy method, for the plate (1) and (2) are supposed functions w_1 and w_2:

$$w_1 = f_1(y)\sin\frac{m\pi x}{a} \qquad (9)$$

$$w_2 = f_2(y)\sin\frac{m\pi x}{a}. \qquad (10)$$

Using (9) and (10), under the load $N_x = \sigma_x \cdot h$, the plates are going to deflect in x direction in the form of *sin* functions, while the functions f_1 (y) and f_2 (y) are giving behaviours of plates (1) and (2) in y direction.

The conditions on edges $x = 0$ and $x = a$ must be satisfied:

$$w_1 = 0\big|_{x=0} \text{ and } D_1\left(\frac{\partial^2 w_1}{\partial x^2}+\nu\frac{\partial^2 w_1}{\partial y^2}\right)=0\Bigg|_{x=a} \qquad (11)$$

$$w_2 = 0\big|_{x=a} \text{ and } D_2\left(\frac{\partial^2 w_2}{\partial x^2}+\nu\frac{\partial^2 w_2}{\partial y^2}\right)=0\Bigg|_{x=a} \qquad (12)$$

which means that plates (1) and (2) are simply supported on $x = 0$ and $x = a$, or the deflections and moments are equal to zero.

Supposed solutions must also satisfy the differential equations for plates (1) and (2), respectively:

$$\left(\frac{1}{4}+\frac{3}{4}\frac{\varphi_k}{\varphi_c}\right)\frac{\partial^4 w_1}{\partial x^4}+2\frac{\partial^4 w_1}{\partial x^2\partial y^2}+\frac{\partial^4 w_1}{\partial y^4}+\frac{N_x\partial w_1}{D_1\partial x^2}=0 \quad (13)$$

$$\left(\frac{1}{4}+\frac{3}{4}\frac{\varphi_k}{\varphi_c}\right)\frac{\partial^4 w_2}{\partial x^4}+2\frac{\partial^4 w_2}{\partial x^2\partial y^2}+\frac{\partial^4 w_2}{\partial y^4}+\frac{N_x\partial w_2}{D_2\partial x^2}=0\,. \quad (14)$$

According to obtained solutions for (13) and (14), we have to find the derivations of (9) and (10):

$$w_1 = f_1(y)f_0(x) = f_1(y)\sin\frac{m\pi x}{a}$$

$$\frac{\partial w_1}{\partial x} = f_1(y)\frac{m\pi}{a}\cos\frac{m\pi x}{a}\ ;\ \frac{\partial^2 w_1}{\partial x^2};\ \frac{\partial^3 w_1}{\partial x^3};\ \frac{\partial^4 w_1}{\partial x^4}$$

$$\frac{\partial^4 w_1}{\partial x^2\partial y^2} = f_1''(y)\left(-\frac{m^2\pi^2}{a^2}\right)\sin\frac{m\pi x}{a}$$

$$\frac{\partial w_1}{\partial y} = f_1'(y)\sin\frac{m\pi x}{a};\ \frac{\partial^2 w_1}{\partial y^2};\ \frac{\partial^3 w_1}{\partial y^3};\ \frac{\partial^4 w_1}{\partial y^4} \qquad (15)$$

$$w_2 = f_2(y)f_0(x) = f_2(y)\sin\frac{m\pi x}{a}$$

$$\frac{\partial w_2}{\partial x} = f_2(y)\frac{m\pi}{a}\cos\frac{m\pi x}{a}\ ;\ \frac{\partial^2 w_2}{\partial x^2};\ \frac{\partial^3 w_2}{\partial x^3};\ \frac{\partial^4 w_2}{\partial x^4}$$

$$\frac{\partial^4 w_2}{\partial x^2\partial y^2} = f_2''(y)\left(-\frac{m^2\pi^2}{a^2}\right)\sin\frac{m\pi x}{a}$$

$$\frac{\partial w_2}{\partial y} = f_2'(y)\sin\frac{m\pi x}{a};\ \frac{\partial^2 w_2}{\partial y^2};\ \frac{\partial^3 w_2}{\partial y^3};\ \frac{\partial^4 w_2}{\partial y^4}\,. \qquad (16)$$

Using (15) and (16) we obtain:

$$\left(\frac{1}{4}+\frac{3}{4}\frac{\varphi_k}{\varphi_c}\right)f_1(y)\frac{m^4\pi^4}{a^4}\sin\frac{m\pi x}{a}+$$

$$+2f_1''(y)\left(-\frac{m^2\pi^2}{a^2}\right)\sin\frac{m\pi x}{a}+f_1^{IV}(y)\sin\frac{m\pi x}{a}+$$

$$+f_1(y)\left(-\frac{m^2\pi^2}{a^2}\right)\sin\frac{m\pi x}{a}=0\,. \qquad (17)$$

At last from (17) we get:

$$f_1^{IV}(y) - 2 f_1''(y)\frac{m^2\pi^2}{a^2} +$$

$$+ f_1(y)\left[\frac{m^4\pi^4}{a^4}\left(\frac{1}{4}+\frac{3}{4}\frac{\varphi_k}{\varphi_c}\right) - \frac{N_x}{D_1}\frac{m^2\pi^2}{a^2}\right] = 0 \,. \quad (18)$$

Partial differential equation of forth order is reduced to ordinal differential equation with constant coefficients of the same order (18).

The solution for (18) is supposed to be:

$$f_1(y) = e^{\lambda y} \,. \quad (19)$$

From (18) we obtain characteristic equation

$$\lambda^4 - 2\frac{m^2\pi^2}{a^2}\lambda^2 + \left[\frac{m^4\pi^4}{a^4}\kappa - k\frac{\pi^4 m^2}{a^2 b^2}\right] \quad (20)$$

where κ_0 and k_1 are:

$$\kappa_0 = \left(\frac{1}{4}+\frac{3}{4}\frac{\varphi_k}{\varphi_c}\right), \text{ and } k_1 = \frac{N_x b^2}{D_1\pi^2} \,. \quad (21)$$

From (20), we get four solutions for λ:

$$\lambda_{1,2,3,4} = \pm\frac{m\pi}{a}\sqrt{1 \pm \sqrt{\left(1+k_1\frac{a^2}{b^2 m^2}\right) - \kappa_0}} \,. \quad (22)$$

As it is $\kappa_0 < 1$, then $\left(1+\frac{k_1 a^2}{b^2 m^2}\right) - \kappa_0 > 0$.

One possible combination for resolving our problem is:

$$\lambda_{1,2} = \pm\frac{m\pi}{a}\sqrt{1 + \sqrt{\left(1+k_1\frac{a^2}{b^2 m^2}\right) - \kappa_0}} = \pm\alpha_1$$

$$\lambda_{3,4} = \pm\,\mathrm{i}\frac{m\pi}{a}\sqrt{-1 + \sqrt{\left(1+k_1\frac{a^2}{b^2 m^2}\right) - \kappa_0}} = \beta_1\cdot\mathrm{i} \,. \quad (23)$$

In this case we obtain our function of deflection:

$$w_1 = \left(C_1 \operatorname{ch}\alpha_1 y + C_2 \operatorname{sh}\alpha_1 y + \right.$$

$$\left. + C_3 \cos\beta_1 y + C_4 \sin\beta_1 y\right)\sin\frac{m\pi x}{a} \,. \quad (24)$$

For the plate (2) we have:

$$\lambda^4 - 2\frac{m^2\pi^2}{a^2}\lambda^2 - \left(\frac{N_x}{D_2}\frac{m^2\pi^2}{a^2} - \frac{m^4\pi^4}{a^4}k_1\right) = 0 \,. \quad (25)$$

For

$$\frac{N_x}{D_2}\frac{m^2\pi^2}{a} = \frac{N_x b^2}{D_1\pi^2}\frac{m^2\pi^4}{b^2}\frac{D_1}{D_2} = k_1\frac{D_1}{D_2}\frac{m^2\pi^4}{b^2} \quad (26)$$

introducing the coefficient ψ:

$$\psi = \frac{D_1}{D_2} \,. \quad (27)$$

The roots of (25) are

$$\lambda'_{1,2,3,4} = \pm\frac{m\pi}{a}\sqrt{1 \pm \sqrt{\left(1+k_1\frac{a^2}{b^2 m^2}\psi\right) - \kappa_0}} \quad (28)$$

or

$$\lambda'_{1,2} = \pm\frac{m\pi}{a}\sqrt{1 \pm \sqrt{\left(1+k_1\frac{a^2}{b^2 m^2}\psi\right) - \kappa_0}} = \pm\alpha_2$$

$$\lambda'_{3,4} = \pm\,\mathrm{i}\frac{m\pi}{a}\sqrt{\sqrt{\left(1+k_1\frac{a^2}{b^2 m^2}\psi\right) - \kappa_0} - 1} = \pm\,\mathrm{i}\beta_2 \,. \quad (29)$$

The solution for the plate (2) is now:

$$w_2 = f_2(y)\sin\frac{m\pi x}{a} = \left(C_5 \operatorname{ch}\alpha_2 y + C_6 \operatorname{sh}\alpha_2 y + \right.$$

$$\left. + C_7 \cos\beta_2 y + C_8 \sin\beta_2 y\right)\sin\frac{m\pi x}{a} \,. \quad (30)$$

Now we must use the boundary conditions (I, II, ..., VII, VIII), in order to get unknown coefficients of (24) and (28):

I condition:

$$w_1 = 0\big|_{y=0} \quad (31)$$

means that deflection for $y = 0$ is zero.

II condition:

$$\left.\frac{\partial^2 w_1}{\partial y^2} + \nu\frac{\partial w_1}{\partial x^2} = 0\,\right|_{y=0} \quad (32)$$

means that the moment is equal to zero.

Using (31) and (32) we get:

$$\text{I: } \left(C_1 \operatorname{ch}0 + C_2 \operatorname{sh}0 + C_3 \cos 0 + C_4 \sin 0\right)\sin\frac{m\pi x}{a} = f_1(0) = 0$$

$$C_1 + C_3 = 0$$

$$\text{II: } w_1 = \left(C_1 \operatorname{ch}\alpha_1 y + C_2 \operatorname{sh}\alpha_1 y + \right.$$

$$\left. + C_3 \cos\beta_1 y + C_4 \sin\beta_1 y\right)\sin\frac{m\pi x}{a}$$

$$\frac{\partial w_1}{\partial y} = \left(\alpha_1 C_1 \operatorname{sh}\alpha_1 y + \alpha_1 C_2 \operatorname{ch}\alpha_1 y - \right.$$

$$\left. - C_3\beta_1 \sin\beta_1 y + C_4\beta_1 \cos\beta_1 y\right)\sin\frac{m\pi x}{a}$$

$$\frac{\partial^2 w_1}{\partial y^2} = \left(\alpha_1^2 C_1 \operatorname{ch}\alpha_1 y + \alpha_1^2 C_2 \operatorname{sh}\alpha_1 y - \right.$$

$$\left. - C_3\beta_1^2 \sin\beta_1 y + C_4\beta_1^2 \cos\beta_1 y\right)\sin\frac{m\pi x}{a} \,. \quad (33)$$

Including (33) into (32) we obtain:

$$\left(C_1\alpha_1^2 \operatorname{ch}0 + C_2\alpha_1^2 \operatorname{sh}0 - C_3\beta_1^2 \cos 0 - C_4\beta_1^2\cdot 0\right)\cdot$$

$$\cdot\sin\frac{m\pi x}{a} + f_1(0)\left(-\frac{m^2\pi^2}{a^2}\nu\sin\frac{m\pi x}{a}\right) = 0 \,. \quad (34)$$

Taking into account (I), $f_1(0) = 0$, condition (II) is reduced to:

$$C_1\alpha_1^2 - C_3\beta_1^2 = 0\,. \qquad (35)$$

Multiplying by α_1^2, and subtracting (I) and (II):

$$-C_1\alpha_1^2 - C_2\alpha_1^2 + C_1\alpha_1^2 + C_3\beta_1^2 = 0$$

$$\left(\alpha_1^2 + \beta_1^2\right)C_3 = 0\,,\ \alpha_1^2 + \beta_1^2 \neq 0 \ \Rightarrow\ C_3 = 0\,. \quad (36)$$

Then from (I):

$$C_1 = -C_3 = 0$$
$$C_1 = C_3 = 0\,. \qquad (37)$$

The function w_1 is now

$$w_1 = \left(C_2 \operatorname{sh}\alpha_1 y + C_4 \sin\beta_2 y\right)\sin\frac{m\pi x}{a} \qquad (38)$$

where α_1 and β_1 are given with (23).

The conditions (III) and (IV) at the end of plate (2) are for $y = b$, the deflection and the moment are also equal to zero.

III condition for plate (2):

$$w_2 = 0\big|_{y=b}\,. \qquad (39)$$

IV condition:

$$D_2\left(\frac{\partial^2 w_2}{\partial y^2} + \nu\frac{\partial^2 w_2}{\partial x^2}\right)\Bigg|_{y=b} = 0\,. \qquad (40)$$

From the function $w_2 = f_2(y)\sin\dfrac{m\pi x}{a}$, we have:

$$w_2 = \left(C_5 \operatorname{ch}\alpha_2 y + C_6 \operatorname{sh}\alpha_2 y + \right.$$
$$\left. + C_7 \cos\beta_2 y + C_8 \sin\beta_2 y\right)\sin\frac{m\pi x}{a}$$

$$\frac{\partial w_2}{\partial y} = \left(C_5\alpha_2 \operatorname{sh}\alpha_2 y + C_6\alpha_2 \operatorname{ch}\alpha_2 y - \right.$$
$$\left. - C_7\beta_2 \sin\beta_2 y + C_8\beta_2 \cos\beta_2 y\right)\sin\frac{m\pi x}{a}$$

$$\frac{\partial^2 w_2}{\partial y^2} = \left(C_5\alpha_2^2 \operatorname{ch}\alpha_2 y + C_6\alpha_2^2 \operatorname{sh}\alpha_2 y - \right.$$
$$\left. - C_7\beta_2^2 \cos\beta_2 y - C_8\beta_2^2 \sin\beta_2 y\right)\sin\frac{m\pi x}{a}\,. \qquad (41)$$

From (III) we obtain:

$$w_2 = f_2(b)\sin\frac{m\pi x}{a}\bigg|_{y=b} = \left(C_5 \operatorname{ch}\alpha_2 b + C_6 \operatorname{sh}\alpha_2 b + \right.$$
$$\left. + C_7 \cos\beta_2 b + C_8 \sin\beta_2 b\right)\sin\frac{m\pi x}{a}$$
$$f_2(b) = 0 = C_5 \operatorname{ch}\alpha_2 b + C_6 \operatorname{sh}\alpha_2 b +$$
$$+ C_7 \cos\beta_2 b + C_8 \sin\beta_2 b = 0 \qquad (42)$$

and from IV:

$$\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\bigg|_{y=b} = 0$$

$$f_2''(b) f_0(x) + \nu f_2(b) f_0''(x) = 0$$

$$w_2 = \left(C_5 \operatorname{ch}\alpha_2 y + C_6 \operatorname{sh}\alpha_2 y + \right.$$
$$\left. + C_7 \cos\beta_2 y + C_8 \sin\beta_2 y\right)\sin\frac{m\pi x}{a}$$

$$\frac{\partial w_2}{\partial y} = \left(C_5\alpha_2 \operatorname{sh}\alpha_2 y + C_6\alpha_2 \operatorname{ch}\alpha_2 y - \right.$$
$$\left. - C_7\beta_2 \sin\beta_2 y + C_8\beta_2 \cos\beta_2 y\right)\sin\frac{m\pi x}{a}$$

$$\frac{\partial^2 w_2}{\partial y^2} = \left(C_5\alpha_2^2 \operatorname{ch}\alpha_2 y + C_6\alpha_2^2 \operatorname{sh}\alpha_2 y - C_7\beta_2^2 \cos\beta_2 y - \right.$$
$$\left. - C_8\beta_2^2 \sin\beta_2 y\right)\sin\frac{m\pi x}{a}\,. \qquad (43)$$

Condition IV gives us:

$$\left(C_5\alpha_2^2 \operatorname{ch}\alpha_2 b + C_6\alpha_2^2 \operatorname{sh}\alpha_2 b - \right.$$
$$\left. - C_7\beta_2^2 \cos\beta_2 b - C_8\beta_2^2 \sin\beta_2 b\right)\cdot$$
$$\cdot\sin\frac{m\pi x}{a} + \nu\left(\frac{m^2\pi^2}{a^2}\right) f_2(b)\sin\frac{m\pi x}{a} = 0\,. \qquad (44)$$

Taking into account condition (III): $f_2(b) = 0$, condition (IV) is reduced to:

$$C_5\alpha_2^2 \operatorname{ch}\alpha_2 b + C_6\alpha_2^2 \operatorname{sh}\alpha_2 b -$$
$$- C_7\beta_2^2 \cos\beta_2 b - C_8\beta_2^2 \sin\beta_2 b = 0\,. \qquad (45)$$

When condition (III) is multiplied first by α_2^2 and then by β_2^2, and in combination with condition (IV), by addition and subtraction we get:

$$C_5\left(\alpha_2^2 + \beta_2^2\right)\operatorname{ch}\alpha_2 b + \left(\alpha_2^2 + \beta_2^2\right)\operatorname{sh}\alpha_2 b = 0$$

$$\alpha_2^2 + \beta_2^2 \neq 0 \ \Rightarrow\ C_5 \operatorname{ch}\alpha_2 b + C_6 \operatorname{sh}\alpha_2 b = 0$$

$$C_6 = -\,C_5\frac{\operatorname{ch}\alpha_2 b}{\operatorname{sh}\alpha_2 b}\,. \qquad (46)$$

$-$ III$\cdot\,\alpha_2^2$ + IV:

$$-C_7\left(\alpha_2^2 + \beta_2^2\right)\cos\beta_2 b - C_8\left(\alpha_2^2 + \beta_2^2\right)\sin\beta_2 b = 0$$

$$C_8 = -\,C_7\frac{\cos\beta_2 b}{\sin\beta_2 b}\,. \qquad (47)$$

When introducing (46) and (47), from (41) we get

$$w_2 = \left[C_5 \operatorname{sh}\alpha_2 y - C_5\frac{\operatorname{ch}\alpha_2 b}{\operatorname{sh}\alpha_2 b}\operatorname{sh}\alpha_2 y + C_7\cos\beta_2 y - \right.$$
$$\left. - C_7\frac{\cos\beta_2 b}{\sin\beta_2 b}\sin\beta_2 y\right]\sin\frac{m\pi x}{a}$$

$$w_2 = C_5\frac{\operatorname{sh}\alpha_2\left(b-y\right)}{\operatorname{sh}\alpha_2 b} + C_7\frac{\sin\beta_2\left(b-y\right)}{\sin\alpha_2 b}\,. \qquad (48)$$

Now, having functions $w_1\,(x,y)$ and $w_2\,(x,y)$:

$$w_1 = f_1(y) f_0(x) = \left(C_2 \operatorname{sh}\alpha_1 y + C_4 \sin\beta_2 y\right) f_0(x)$$

$$w_2 = f_2(y) f_0(x) =$$

$$= \left(C_5 \frac{\operatorname{ch}\alpha_2 (b-y)}{\operatorname{sh}\alpha_2 b} + C_7 \frac{\sin\beta_2 (b-y)}{\sin\alpha_2 b}\right) f_0(x) \,. \quad (49)$$

Now, we use the condition at $y = \eta$. Along the edge $y = \eta$, deflections between (1) and (2) are equal

$$w_1(x,y)\big|_{y=\eta} = w_2(x,y)\big|_{y=\eta} \,. \quad (50)$$

The condition (50) is reduced to:

$$f_1(y)\big|_{y=\eta} = f_2(y)\big|_{y=\eta}$$

$$C_2 \operatorname{sh}\alpha_1 \eta + C_4 \sin\beta_1 \eta =$$

$$C_5 \frac{\operatorname{sh}\alpha_2 (b-\eta)}{\operatorname{sh}\alpha_2 b} + C_7 \frac{\sin\beta_2 (b-\eta)}{\sin\beta_2 b} \,, \quad (51)$$

or

$$C_2 \operatorname{sh}\alpha_1 \eta + C_4 \sin\beta_1 \eta - C_5 \frac{\operatorname{sh}\alpha_2 (b-\eta)}{\operatorname{sh}\alpha_2 b} -$$

$$-C_7 \frac{\sin\beta_2 (b-\eta)}{\sin\beta_2 b} = 0 \,. \quad (52)$$

Condition VI: We need the derivatives of (49):

$$\frac{\partial w_1}{\partial y} = \left(C_2 \alpha_1 \operatorname{ch}\alpha_1 y + C_4 \beta_1 \cos\beta_1 y\right) \sin\frac{m\pi x}{a}$$

$$\frac{\partial^2 w_1}{\partial y^2} = \left(C_2 \alpha_1^2 \operatorname{sh}\alpha_1 y - C_4 \beta_1^2 \sin\beta_1 y\right) \sin\frac{m\pi x}{a}$$

$$\frac{\partial^3 w_1}{\partial y^3} = \left(C_2 \alpha_1^3 \operatorname{ch}\alpha_1 y - C_4 \beta_1^3 \cos\beta_1 y\right) \sin\frac{m\pi x}{a}$$

$$\frac{\partial w_2}{\partial y} = \left(-C_5 \alpha_2 \frac{\operatorname{ch}_2 (b-y)}{\operatorname{sh}\alpha_2 b} - C_7 \beta_2 \frac{\cos\beta_2 (b-y)}{\sin\beta_2 b}\right) \sin\frac{m\pi x}{a}$$

$$\frac{\partial^2 w_2}{\partial y^2} = \left(C_5 \alpha_2^2 \frac{\operatorname{sh}\alpha_2 (b-y)}{\operatorname{sh}\alpha_2 b} - C_7 \beta_2^2 \frac{\sin\beta_2 (b-y)}{\sin\beta_2 b}\right) \sin\frac{m\pi x}{a}$$

$$\frac{\partial^3 w_2}{\partial y^3} = \left(-C_5 \alpha_2^3 \frac{\operatorname{ch}\alpha_2 (b-y)}{\operatorname{sh}\alpha_2 b} + \right.$$

$$\left. + C_7 \beta_2^3 \frac{\cos\beta_2 (b-y)}{\sin\beta_2 b}\right) \sin\frac{m\pi x}{a} \,. \quad (53)$$

Condition VI is: the slope of plate (1) is equal to the slope of plate (2) for $y = \eta$:

$$\left.\frac{\partial w_1(x,y)}{\partial y}\right|_{y=\eta} = \left.\frac{\partial w_2(x,y)}{\partial y}\right|_{y=\eta} \,, \quad (54)$$

this means that using (53) we come to $f_1'(\eta) = f_2'(\eta)$, or

$$C_2 \alpha_1 \operatorname{ch}\alpha_1 \eta + C_4 \beta_1 \cos\beta_1 \eta + C_5 \alpha_2 \frac{\operatorname{ch}\alpha_2 (b-\eta)}{\operatorname{sh}\alpha_2 b} +$$

$$+ C_7 \beta_2 \frac{\cos\beta_2 (b-\eta)}{\sin\beta_2 b} = 0 \,. \quad (55)$$

Condition VII: Along the line of discontinuity, the moment of plate (1) is equal to the moment of plate (2), for $y = \eta$:

$$D_1 \left.\left(\frac{\partial^2 w_1}{\partial y^2} + \nu \frac{\partial^2 w_1}{\partial x^2}\right)\right|_{y=\eta} = D_2 \left.\left(\frac{\partial^2 w_2}{\partial y^2} + \nu \frac{\partial^2 w_2}{\partial x^2}\right)\right|_{y=\eta} . \quad (56)$$

Using (53), and notation $\frac{D_1}{D_2} = \psi$, and having derivatives we get:

$$\psi \left[f_1''(y) - \nu \frac{m^2 \pi^2}{a^2} f_1(y) \right] \sin\frac{m\pi x}{a} =$$

$$= \left[f_2''(y) - \nu \frac{m^2 \pi^2}{a^2} f_2(y) \right] \sin\frac{m\pi x}{a} \quad (57)$$

or

$$\psi \left[\left(\alpha_1^2 C_2 \operatorname{sh}\alpha_1 \eta - C_4 \beta_1^2 \sin\beta_1 \eta\right) - \right.$$

$$\left. - \nu \frac{m^2 \pi^2}{a^2} \left(C_2 \operatorname{sh}\alpha_1 \eta + C_4 \sin\beta_1 \eta\right) \right] =$$

$$= \left[C_5 \alpha_2^2 \frac{\operatorname{sh}\alpha_2 (b-\eta)}{\operatorname{sh}\alpha_2 b} - C_7 \beta_2^2 \frac{\sin\beta_2 (b-\eta)}{\sin\beta_2 b} \right] -$$

$$- \nu \frac{m^2 \pi^2}{a^2} \left[C_5 \frac{\operatorname{sh}\alpha_2 (b-\eta)}{\operatorname{sh}\alpha_2 b} - C_7 \frac{\sin\beta_2 (b-\eta)}{\sin\beta_2 b} \right] . \quad (58)$$

The final form is:

$$C_2 \psi \left(\alpha_1^2 - \nu \frac{m^2 \pi^2}{a^2} \right) \operatorname{sh}\alpha_1 \eta - C_4 \psi \left(\beta_1^2 + \nu \frac{m^2 \pi^2}{a^2} \sin\beta_1 \eta \right) -$$

$$- C_5 \left(\alpha_2^2 - \nu \frac{m^2 \pi^2}{a^2} \right) \frac{\operatorname{sh}\alpha_2 (b-\eta)}{\operatorname{sh}\alpha_2 b} +$$

$$+ C_7 \left(\beta_2^2 + \nu \frac{m^2 \pi^2}{a^2} \right) \frac{\sin\beta_2 (b-\eta)}{\sin\beta_2 b} \,. \quad (59)$$

Condition VIII: Sharing forces along the contact of plate (1) and (2), for $y = \eta$ must be equal:

$$D_1 \left.\left[\frac{\partial^3 w_1}{\partial y^3} + (2-\nu) \frac{\partial^3 w_1}{\partial x^2 \partial y} \right]\right|_{y=\eta} =$$

$$= D_2 \left.\left[\frac{\partial^3 w_2}{\partial y^3} + (2-\nu) \frac{\partial^3 w_3}{\partial x^2 \partial y} \right]\right|_{y=\eta} \quad (60)$$

which can be written, introducing ψ, in the form

$$\psi \left.\left[f_1'''(y) - (2-\nu) \frac{m^2 \pi^2}{a^2} f_1'(y) \right]\right|_{y=\eta} \sin\frac{m\pi x}{a} =$$

$$= \left[f_2'''(y) - (2-\nu) \frac{m^2 \pi^2}{a^2} f_2'(y) \right] \sin\frac{m\pi x}{a} \quad (61)$$

or, using necessary derivatives (16), for (60), we obtain:

$$\psi\left\{\left[C_2\alpha_1^3\,\mathrm{ch}\alpha_1\eta - C_4\beta_1^3\cos\beta_1\eta\right]-(2-\nu)\frac{m^2\pi^2}{a^2}\cdot\right.$$

$$\left.\cdot\left[\alpha_1 C_2\,\mathrm{ch}\alpha_1\eta+\beta_1 C_4\cos\beta_1\eta\right]\right\}\sin\frac{m\pi x}{a}=$$

$$=\left\{\left[-C_5\alpha_2^3\frac{\mathrm{ch}\alpha_2(b-\eta)}{\mathrm{sh}\alpha_2 b}+C_7\beta_2^3\frac{\cos\beta_2(b-\eta)}{\sin\beta_2 b}\right]-(2-\nu)\cdot\right.$$

$$\left.\cdot\frac{m^2\pi^2}{a^2}\left[-C_5\alpha_2\frac{\mathrm{ch}\alpha_2(b-\eta)}{\mathrm{sh}\alpha_2\eta}-C_7\beta_2\frac{\cos\beta_2(b-\eta)}{\sin\beta_2 b}\right]\right\}\cdot$$

$$\cdot\sin\frac{m\pi x}{a}. \qquad (62)$$

Form (62) can be written

$$\psi\left\{C_2\left[\alpha_1^3-(2-\nu)\alpha_1\frac{m^2\pi^2}{a^2}\right]\mathrm{ch}\alpha_1\eta-\right.$$

$$\left.-C_4\left[\beta_1^3+(2-\nu)\beta_1\frac{m^2\pi^2}{a^2}\right]\cos\beta_1\eta\right\}=$$

$$=-C_5\left[\alpha_2^3-(2-\nu)\alpha_2\frac{m^2\pi^2}{a^2}\right]\frac{\mathrm{ch}\alpha_2(b-\eta)}{\mathrm{sh}\alpha_2 b}+$$

$$+C_7\left[\beta_2^3+(2-\nu)\beta_2\frac{m^2\pi^2}{a^2}\right]\frac{\cos\beta_2(b-\eta)}{\sin\beta_2 b} \qquad (63)$$

or

$$\psi C_2\left[\alpha_1^3-(2-\nu)\alpha_1\frac{m^2\pi^2}{a^2}\right]\mathrm{ch}\alpha_1\eta-$$

$$-\psi C_4\left[\beta_1^3+(2-\nu)\beta_1\frac{m^2\pi^2}{a^2}\right]\cos\beta_1\eta+$$

$$+C_5\left[\alpha_2^3-(2-\nu)\alpha_2\frac{m^2\pi^2}{a^2}\right]\frac{\mathrm{ch}\alpha_2(b-\eta)}{\mathrm{sh}\alpha_2 b}-$$

$$-C_7\left[\beta_2^3+(2-\nu)\beta_2\frac{m^2\pi^2}{a^2}\right]\frac{\cos\beta_2(b-\eta)}{\sin\beta_2 b}=0. \qquad (64)$$

We have obtained 4 homogeneous algebraic equations with unknown constants C_2, C_4, C_5 and C_7. As those systems have trivial solution when all constants are equal to zero, the solution could be obtained only in the case when the determinant of the system (65) is equal to zero.

3. CONCLUSION

The determinant (65) has given solutions for critical coefficient k for different ratios of a/b (Fig. 3), for 3 cases: m = 1, 2 and 3. Minimum value of buckling force is obtained for m = 1 and a/b = 0.8 using the relation

$$k_1=\frac{(N_x)_{\mathrm{cri}}}{D_1}\frac{b^2}{\pi^2}=\frac{(\sigma_x)_{\mathrm{cri}}h_1 b^2}{D_1\pi^2} \qquad (66)$$

	C_2	C_4	C_5	C_7	
V	$\mathrm{ch}\alpha_1\eta$	$\sin\beta_1\eta$	$\frac{\mathrm{sh}\alpha_2(b-\eta)}{\mathrm{sh}\alpha_2 b}$	$\frac{\sin\beta_2(b-\eta)}{\sin\beta_2 b}$	
VI	$\alpha_1\mathrm{ch}\alpha_1\eta$	$\beta_1\cos\beta_1\eta$	$\alpha_2\frac{\mathrm{ch}\alpha_2(b-\eta)}{\mathrm{sh}\alpha_2 b}$	$\beta_2\frac{\cos\beta_2(b-\eta)}{\sin\beta_2 b}$	=0
VII	$\psi\left(\alpha_1^2-\nu\frac{m^2\pi^2}{a^2}\right)\mathrm{sh}\alpha_1\eta$	$-\psi\left(\beta_1^2+\nu\frac{m^2\pi^2}{a^2}\right)\sin\beta_1\eta$	$\left(\alpha_2^2-\nu\frac{m^2\pi^2}{a^2}\right)\frac{\mathrm{sh}\alpha_2(b-\eta)}{\mathrm{sh}\alpha_2 b}$	$\left(\beta_2^2-\nu\frac{m^2\pi^2}{a^2}\frac{\sin\beta_2(b-\eta)}{\sin\beta_2 b}\right)$	
VIII	$\psi\left[\alpha_1^3-\alpha_1(2-\nu)\frac{m^2\pi^2}{a^2}\right]\mathrm{ch}\alpha_1\eta$	$-\psi\left[\beta_1^3-\beta_1(2-\nu)\frac{m^2\pi^2}{a^2}\cos\alpha_1\eta\right]$	$\left[\alpha_2^3-\alpha_2(2-\nu)\frac{m^2\pi^2}{a^2}\right]\frac{\mathrm{ch}\alpha_2(b-\eta)}{\mathrm{sh}\alpha_2 b}$	$-\left[\beta_2^3+\beta_2(2-\nu)\frac{m^2\pi^2}{a^2}\frac{\cos\beta_2(b-\eta)}{\sin\beta_2 b}\right]$	

C_2	C_4	C_5	C_7	
$\mathrm{ch}\alpha_1\eta$	$\sin\beta_1\eta$	$-\mathrm{sh}\alpha_2(b-\eta)$	$-\sin\beta_2(b-\eta)$	
$\alpha_1\mathrm{ch}\alpha_1\eta$	$\beta_1\cos\beta_1\eta$	$\alpha_2\mathrm{ch}\alpha_2(b-\eta)$	$\beta_2\cos\beta_2(b-\eta)$	=0. (65)
$\psi\left(\alpha_1^2-\nu\frac{m^2\pi^2}{a^2}\right)\mathrm{sh}\alpha_1\eta$	$-\psi\left(\beta_1^2+\nu\frac{m^2\pi^2}{a^2}\right)\sin\beta_1\eta$	$-\left(\alpha_2^2-\nu\frac{m^2\pi^2}{a^2}\right)\mathrm{sh}\alpha_2(b-\eta)$	$\left(\beta_2^2+\nu\frac{m^2\pi^2}{a^2}\right)\sin\beta_2(b-\eta)$	
$\psi\left[\alpha_1^3-\alpha_1(2-\nu)\frac{m^2\pi^2}{a^2}\right]\mathrm{ch}\alpha_1\eta$	$-\psi\left[\beta_1^3+\beta_1(2-\nu)\frac{m^2\pi^2}{a^2}\cos\alpha_1\eta\right]$	$\left[\alpha_2^3-\alpha_2(2-\nu)\frac{m^2\pi^2}{a^2}\right]\mathrm{ch}\alpha_2(b-\eta)$	$-\left[\beta_2^3+\beta_2(2-\nu)\frac{m^2\pi^2}{a^2}\right]\cos\beta_2(b-\eta)$	

or the critical stress in the plate (1)

$$(\sigma_x)_{\text{cri}} = \frac{D_1 \pi^2 k_1}{h_1 b^2}, \tag{67}$$

where $D_1 = \dfrac{E h_1^3}{12(1-\nu)}$ for aluminium.

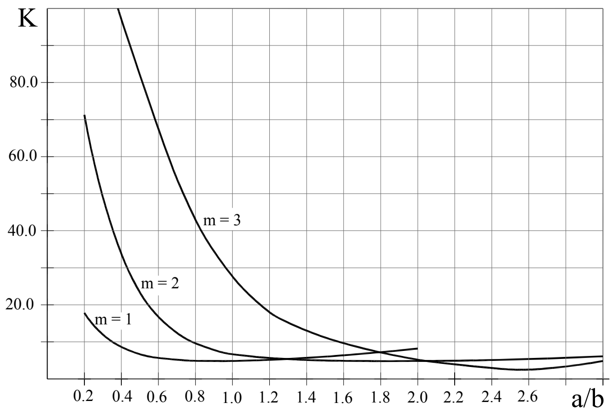

Figure 3. Final solution for critical force as a function of (*a*/*b*)

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ПРОБЛЕМ СТАБИЛНОСТИ СТЕПЕНАСТИХ ПЛОЧА У ПЛАСТИЧНОЈ ОБЛАСТИ

Момчило Дуњић

У области стабилности правоугаоних степенастих плоча, у последње време, обрађени су многи проблеми у еластичној области. Та проблематика још није обрађивана у пластичној области, било у области малих еласто-пластичних деформација, било у области течења. У овом раду се говори о проблему губитка стабилности степенасте правоугаоне плоча (са две дебљине) у пластичној зони деформисања, оптерећене притискујућом силом у равни плоче која делује дуж ивица по којима се мења дебљина.