

An Application of Continuum Theory on the Case of One-Dimensional Sedimentation

Boško P. Cvetković

PhD Student
University of Belgrade
Faculty of Mechanical Engineering

Dragoslav S. Kuzmanović

Full Professor
University of Belgrade
Faculty of Traffic and Transport Engineering

Predrag A. Cvetković

Full Professor
University of Belgrade
Faculty of Traffic and Transport Engineering

The fundamental balance laws of the fluid phase and the disperse phase of suspension are presented. Constitutive equations are developed and the method of Lagrangian multipliers is used. The objective of this paper is to use the continuum theory of two-phase flow to model the one-dimensional sedimentation of particle through a fluid in the direction of vertical axis. After using the basic equations of motion and corresponding restrictions, the diffusion equation of quasi-steady sedimentation is derived. The fact that the diffusion equation is parabolic, when a diffusivity term is included, suggests the possibility that sedimentation problem could be solved numerically and then graphically presented.

Keywords: continuum theory, suspension, constitutive equations, volume distribution function, free energy, drag coefficient.

1. INTRODUCTION

The sedimentation of particles in liquid is of interest in many chemical engineering processes. The variables associated with the liquid and the particles can be introduced as continuous field from the beginning, and balance equations and constitutive relations can be postulated [1-5]. In paper [6] the equations for two-phase flow are used to analyze the one-dimensional sedimentation of solid particles in a stationary container of liquid. A derivation of the equations of motion is presented upon Hamilton's external variation principle. The same result was obtained using the continuum theory for suspension flow presented in [7].

In paper [7-10] the continuum theory for suspension flow is developed. Fluid, the basic phase of suspension, and the disperse phase are constituents of the mixture. Using the basic balance laws of continuum mechanics and the method of Lagrangian multipliers, a set of constitutive equations for saturated suspension is deduced.

In this paper, the basic equations of motion of different phases are derived. The model of one-dimensional sedimentation of disperse phase is considered. Using the basic equations of motion and corresponding restrictions, the diffusion equation of quasi-steady sedimentation is derived. This result is the same as in paper [6].

2. BASIC LAWS OF MOTION

The basic phase (fluid) and the disperse phase (constituents) of suspension must satisfy the basic laws of motion of continuum mechanics [7]. In the absence of chemical reaction between the phases and various constituents, the following field equations must hold:

Conservation of mass

$$\frac{\partial \rho^r}{\partial t} + (\rho^r v_i^r)_{,i} = 0, \quad (1)$$

where $r = f$ or α for fluid and α -th constituent of disperse phase of suspension.

Balance of momentum

$$\rho^r \frac{d v_i^r}{d t} = t_{ij,j}^r + \rho^r f_i^r + P_i^r, \quad (2)$$

where t_{ij}^r , f_i^r and P_i^r are stress tensor, body force per unit mass and internal body force, respectively.

Balance of moment of momentum

$$t_{[jk]}^r = 0. \quad (3)$$

Balance of equilibrated force

$$\rho^r \frac{d}{d t} (k^r \dot{n}^r) = h_{i,i}^r + \rho^r l^r + \hat{g}^r, \quad (4)$$

where k^r , n^r , h_i^r , l^r and \hat{g}^r are equilibrated force, the volume distribution function (concentration), equilibrated stress vector, equilibrated force per unit mass and internal equilibrated force, respectively.

Balance of energy

$$\rho^r \dot{e}^r = t_{ki}^r v_{i,k}^r + h_k^r \dot{n}_{,k}^r + q_{k,k}^r + \rho^r r^r - \hat{g}^r \dot{n}^r - P_k^r u_k^r, \quad (5)$$

where q_k^r and r^r are heat flux vector and internal heat source per unit mass.

The following entropy inequality is postulated

$$\rho^f \dot{\eta}^f - \left(\frac{q_k^f}{\theta^f} \right)_{,k} - \frac{\rho^f r^f}{\theta^f} + \sum_{\alpha=1}^N \left[\rho^\alpha \dot{\eta}^\alpha - \left(\frac{q_k^\alpha}{\theta^\alpha} \right)_{,k} - \frac{\rho^\alpha r^\alpha}{\theta^\alpha} \right] \geq 0, \quad (6)$$

Received: March 2011, Accepted: May 2011

Correspondence to: Boško Cvetković
Faculty of Mechanical Engineering,
Kraljice Marije 16, 11120 Belgrade 35, Serbia
E-mail: boskocvetkovic@gmail.com

where η^r is entropy density per unit mass, θ^f and θ^α is temperature for fluid and α -th constituent of disperse phase of suspension.

Using the free energies of phases

$$\psi^r = e^r - \eta^r \theta^r, \quad (6a)$$

the constrains and the method of Lagrangian multipliers λ for the case of fully saturated suspension flows, the entropy inequality is used in the form

$$\begin{aligned} & \frac{1}{\theta^f} \left[-\rho^f (\dot{\psi}^f + \eta^f \dot{\theta}^f) + \frac{q_k^f \theta_k^f}{\theta^f} + t_{kl}^f d_{lk}^f + h_k^f \dot{n}_{,k}^f + \right. \\ & \quad \left. + \lambda^f n^f v_{k,k}^f - (\hat{g}^f - \lambda \theta^f - \lambda^f) \dot{n}^f - \right. \\ & \quad \left. - (P_k^f + \lambda \theta^f n_{,k}^f) u_k^f \right] + \\ & \quad + \sum_{\alpha=1}^N \frac{1}{\theta^\alpha} \left[-\rho^\alpha (\dot{\psi}^\alpha + \eta^\alpha \dot{\theta}^\alpha) + \frac{q_k^\alpha \theta_k^\alpha}{\theta^\alpha} + t_{kl}^\alpha d_{lk}^\alpha + \right. \\ & \quad \left. + h_k^\alpha \dot{n}_{,k}^\alpha + \lambda^\alpha n^\alpha v_{k,k}^\alpha - (\hat{g}^\alpha - \lambda \theta^\alpha - \lambda^\alpha) \dot{n}^\alpha - \right. \\ & \quad \left. - (P_k^\alpha + \lambda \theta^\alpha n_{,k}^\alpha) u_k^\alpha \right] \geq 0. \quad (7) \end{aligned}$$

3. CONSTITUTIVE EQUATIONS

Let the sets $\{A^f\}$ and $\{A^\alpha\}$ be the set of independent constitutive variables of the basic and disperse phases of suspension. The following independent variables are postulated

$$\begin{aligned} \{A^f\} & \equiv \{\rho^f; n^f; n_i^f; \dot{n}^f; \dot{n}_i^f; d_{ij}^f; \theta^f; \theta_i^f\} \\ \{A^\alpha\} & \equiv \{\rho^\alpha; n^\alpha; n_i^\alpha; \dot{n}^\alpha; \dot{n}_i^\alpha; d_{ij}^\alpha; \theta^\alpha; \theta_i^\alpha\}. \quad (8) \end{aligned}$$

Using the principle of phase separation by Drew and Segel [2], after lengthy calculation, we find the following set of constitutive equations for the stress tensors, equilibrated stress vectors, the internal forces and the internal equilibrated forces of the basic phase and disperse phase of suspension, i.e.

$$E^r_{kl} = -(\pi^r + \lambda^r n^r) \delta_{kl} - 2\alpha^r (n^r) n_{,k}^r n_{,l}^r, \quad (9)$$

$$E^r h_k^r = 2\alpha^r (n^r) n_{,k}^r, \quad (10)$$

$$P_k^f = \sum_{\alpha=1}^N K^\alpha (v_k^\alpha - v_k^f) - \bar{\lambda} n_{,k}^f, \quad (11)$$

$$P_k^\alpha = K^\alpha (v_k^\alpha - v_k^f) - \bar{\lambda} n_{,k}^\alpha, \quad (12)$$

$$\begin{aligned} \hat{g}^r & = -a_0^r n^r - n^r \frac{d}{dn^r} \left(\frac{\alpha^r}{n^r} \right) n_{,k}^r n_{,k}^r + \\ & \quad + \pi^r + \bar{\lambda} + \lambda^r - G^r \dot{n}^r, \quad (13) \end{aligned}$$

where, as well known quantities, K^α are the drag coefficients of α -th particulate constituent, α^r are positive material constants, π^r are the thermodynamic pressures and G^r are equilibrated drag coefficients and $\bar{\lambda} = \lambda \theta$.

4. GENERAL EQUATIONS OF MOTION

Direct substitution of the constitutive equations for the stress vectors, equilibrated stress vectors, the internal forces and the internal equilibrated forces of the basic phase and the disperse phase into the equations of balance (1) to (4), yield the general equations of motion:

Momentum of the basic phase

$$\begin{aligned} \rho^f \frac{dv_i^f}{dt} & = -\pi_{,i}^f - 2(\alpha^f n_{,k}^f n_{,l}^f)_{,k} + \\ & \quad + \sum_{\alpha} K^\alpha (v_k^\alpha - v_k^f) - \bar{\lambda} n_{,k}^f + \rho^f f_i^f. \quad (14) \end{aligned}$$

Momentum of the disperse phase

$$\begin{aligned} \rho^\alpha \frac{dv_i^\alpha}{dt} & = -\pi_{,i}^\alpha - 2(\alpha^\alpha n_{,k}^\alpha n_{,l}^\alpha)_{,k} + \\ & \quad + \sum_{\alpha} K^\alpha (v_k^f - v_k^\alpha) - \bar{\lambda} n_{,k}^\alpha + \rho^\alpha f_i^\alpha. \quad (15) \end{aligned}$$

Equilibrated force of the basic phase and the disperse phase

$$\begin{aligned} \rho^r \frac{d}{dt} (k^r \dot{n}^r) & = 2(\alpha^r n_{,k}^r)_{,k} + (\xi_V^r \dot{n}_{,k}^r)_{,k} + \rho^r l^r - \\ & \quad - a_0^r n^r - n^r \frac{d}{dt} \left(\frac{\alpha^r}{n^r} \right) n_{,k}^r n_{,k}^r + \bar{\lambda} + \lambda^r - G^r \dot{n}^r. \quad (16) \end{aligned}$$

Independent variation of variables \dot{n}^r , $\dot{n}_{,k}^r$ and ρ^r appear linearly in (16) and then their coefficients can be neglected. Then it follows that

$$k^r \equiv 0, \quad l^r \equiv 0, \quad (16a)$$

and also

$$\xi_V^r \equiv 0, \quad G^r \equiv 0, \quad (16b)$$

then (16) may be solved for Lagrangian multipliers, i.e.

$$\lambda^r = -2\alpha^r n_{,kk}^r - \left(\frac{d\alpha^r}{dn^r} + \frac{\alpha^r}{n^r} \right) n_{,k}^r n_{,k}^r - \bar{\lambda} + a_0^r n^r. \quad (17)$$

For Lagrangian multipliers λ^r we find $\lambda^r = \pi^r/n^r$.

If the thermodynamic pressures π^f and π^α are replaced by $\lambda^f n^f$ and $\lambda^\alpha n^\alpha$ in (14) and (15), then, by using (17), these equations become

$$\begin{aligned} \rho_0^f n^f \frac{dv_i^f}{dt} & = 2\alpha^f n^f n_{,kk}^f + 2n^f \frac{d\alpha^f}{dn^f} (n_{,l}^f n_{,kk}^f + \\ & \quad + n_{,k}^f n_{,kl}^f) + n^f \frac{d^2 \alpha^f}{d(n^f)^2} n_{,k}^f n_{,k}^f n_{,l}^f - 2a_0^f n^f n_{,l}^f + \bar{\lambda}_l n^f + \\ & \quad + \sum_{\alpha} K^\alpha (v_l^\alpha - v_l^f) + \rho_0^f n^f b_l^f, \quad (18) \end{aligned}$$

$$\begin{aligned} \rho_0^\alpha n^\alpha \frac{dv_i^\alpha}{dt} &= 2\alpha^\alpha n^\alpha n_{,kk}^\alpha + 2n^\alpha \frac{d\alpha^\alpha}{dn^\alpha} \left(n_{,l}^\alpha n_{,kk}^\alpha + \right. \\ &+ n_{,k}^\alpha n_{,kl}^\alpha \left. \right) + n^\alpha \frac{d^2\alpha^\alpha}{d(n^\alpha)^2} n_{,k}^\alpha n_{,k}^\alpha n_{,l}^\alpha - 2a_0^\alpha n^\alpha n_{,l}^\alpha + \bar{\lambda}_{,l} n^\alpha + \\ &+ K^\alpha \left(v_l^f - v_l^\alpha \right) + \rho_0^\alpha n^\alpha b_l^\alpha. \end{aligned} \quad (19)$$

5. THE ONE-DIMENSIONAL SEDIMENTATION

We apply the general equations of motion (18) and (19) to suspension flow. For the reason of simplicity, we suppose that the material constants α^r and a_0^r are independent of the fluid volume distribution function and volume distribution function of α -th particulate constituent. In this case (Fig. 1), the equations of motion become:

Mass

$$\frac{\partial n^r}{\partial t} + \nabla \left(n^r v_i^r \right) = 0. \quad (20)$$

Momentum

$$\begin{aligned} \rho_0^f \frac{dv_i^f}{dt} &= \nabla \bar{\lambda} + 2\alpha^f \nabla \nabla^2 n^f - 2a_0^f \nabla n^f + \\ &+ \sum_{\alpha} \frac{K^\alpha}{n_0^f} \left(v_l^\alpha - v_l^f \right) + \rho_0^f b_l, \end{aligned} \quad (21)$$

$$\begin{aligned} \rho_0^\alpha \frac{dv_i^\alpha}{dt} &= \nabla \bar{\lambda} + 2\alpha^\alpha \nabla \nabla^2 n^\alpha - 2a_0^\alpha \nabla n^\alpha + \\ &+ \frac{K^\alpha}{n_0^\alpha} \left(v_l^f - v_l^\alpha \right) + \rho_0^\alpha b_l, \end{aligned} \quad (22)$$

where it is assumed that the body force accelerations are the same, i.e. $b_l^f = b_l^\alpha = b_l$.

If we keep a record, as pointed out in the introduction that the one-dimensional sedimentation in the x direction will be considered, then the equations of masses are given by

$$\frac{\partial n^r}{\partial t} + \frac{\partial}{\partial x} \left(n^r v_i^r \right) = 0. \quad (23)$$

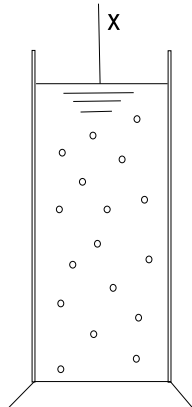


Figure 1. Sedimentation configuration

Neglecting derivatives of higher order and using the total time derivatives for velocities, equations of momentum in one-dimensional form are

$$\begin{aligned} \rho_0^f \left(\frac{\partial v_i^f}{\partial t} + v_k^f \frac{\partial v_i^f}{\partial x} \right) &= \nabla \bar{\lambda} - 2a_0^f \frac{\partial n^f}{\partial x} + \\ &+ \sum_{\alpha} \frac{K^\alpha}{n_0^f} \left(v_l^\alpha - v_l^f \right) + \rho_0^f b_l, \end{aligned} \quad (24)$$

$$\begin{aligned} \rho_0^\alpha \left(\frac{\partial v_i^\alpha}{\partial t} + v_k^\alpha \frac{\partial v_i^\alpha}{\partial x} \right) &= \nabla \bar{\lambda} - 2a_0^\alpha \frac{\partial n^\alpha}{\partial x} + \\ &+ \frac{K^\alpha}{n_0^\alpha} \left(v_l^f - v_l^\alpha \right) + \rho_0^\alpha b_l. \end{aligned} \quad (25)$$

The linearized form of the equations of mass conservations (20) are given by

$$\frac{\partial n^r}{\partial t} + n_0^r \nabla v_i^r = 0. \quad (26)$$

Summing (26) for all species and using the saturation condition given by $n^f = 1 - n$ ($n = \sum_{\alpha} n^\alpha$),

we find

$$\nabla \left(n_0^f v_i^f + \sum_{\alpha} n_0^\alpha v_i^\alpha \right) = 0, \quad (27)$$

from where it is

$$v_i^f = -\frac{1}{n_0^f} \sum_{\alpha} n_0^\alpha v_i^\alpha, \quad (28)$$

where integration constant C can be neglected.

Eliminating $\nabla \bar{\lambda}$ between (24) and (25) and using (28), we find

$$\begin{aligned} \rho_0^\alpha \frac{\partial v_i^\alpha}{\partial t} + \rho_0^\alpha v_k^\alpha \frac{\partial v_i^\alpha}{\partial x} + \frac{\rho_0^f}{n_0^f} \sum_{\alpha} n_0^\alpha \frac{\partial v_i^\alpha}{\partial t} - \\ - \frac{\rho_0^f}{(n_0^f)^2} \sum_{\alpha} n_0^\alpha v_i^\alpha \sum_{\beta} n_0^\beta \frac{\partial v_i^\beta}{\partial x} + 2 \sum_{\alpha} a_0^\alpha \frac{\partial n^\alpha}{\partial x} = \\ = -2 \sum_{\alpha} \frac{K^\alpha}{n_0^\alpha n_0^f} \left(n_0^f v_i^\alpha + \sum_{\beta} n_0^\beta v_i^\beta \right) + \\ + \left(\rho_0^f - \rho_0^\alpha \right) b_i. \end{aligned} \quad (29)$$

The linearized form of (29) and (23) is

$$\begin{aligned} \rho_0^\alpha \frac{\partial v_i^\alpha}{\partial t} + \frac{\rho_0^f}{n_0^f} \sum_{\alpha} n_0^\alpha \frac{\partial v_i^\alpha}{\partial t} + 2 \sum_{\alpha} a_0^\alpha \frac{\partial n^\alpha}{\partial x} = \\ = -2 \sum_{\alpha} \frac{K^\alpha}{n_0^\alpha n_0^f} \left(n_0^f v_i^\alpha + \sum_{\beta} n_0^\beta v_i^\beta \right) + \\ + \left(\rho_0^f - \rho_0^\alpha \right) b_i, \end{aligned} \quad (30)$$

$$\frac{\partial n^\alpha}{\partial t} + n_0^\alpha \frac{\partial v_i^\alpha}{\partial t} = 0. \quad (31)$$

Eliminating v_i^α between (30) and (31), yields the single equation for n^α , i.e.

$$\begin{aligned} -\frac{\rho_0^\alpha}{n_0^\alpha} \frac{\partial^2 n^\alpha}{\partial t^2} - \frac{\rho_0^f}{n_0^f} \sum_\alpha \frac{n_0^f}{n_0^\alpha} \frac{\partial^2 n^\alpha}{\partial t^2} + 2 \sum_\alpha a_0^\alpha \frac{\partial^2 n^\alpha}{\partial x^2} = \\ = 2 \sum_\alpha \frac{K^\alpha}{(n_0^\alpha)^2} \frac{\partial n^\alpha}{\partial t} + 2 \sum_\alpha \frac{K^\alpha}{n_0^\alpha n_0^f} \sum_\beta \frac{\partial n^\beta}{\partial t}. \end{aligned} \quad (32)$$

Then, if the acceleration terms in (29) are neglected (the case of quasi-steady sedimentation), (32) becomes the generalized diffusion equation

$$\sum_\alpha a_0^\alpha \frac{\partial^2 n^\alpha}{\partial x^2} = \sum_\alpha \frac{K^\alpha}{(n_0^\alpha)^2} \frac{\partial n^\alpha}{\partial t} + \sum_\alpha \frac{K^\alpha}{n_0^\alpha n_0^f} \sum_\beta \frac{\partial n^\beta}{\partial t}, \quad (33)$$

i.e.

$$a_0^\alpha \frac{\partial^2 n}{\partial x^2} = \frac{K^\alpha}{(n_0^\alpha)^2} \frac{\partial n^\alpha}{\partial t} + \frac{K^\alpha}{n_0^\alpha n_0^f} \sum_\beta \frac{\partial n^\beta}{\partial t}. \quad (34)$$

For the case of simple suspension [3]

$$\sum_\beta \frac{\partial n^\beta}{\partial t} = 0, \quad (34a)$$

we obtain the diffusion equation

$$\frac{\partial n^\alpha}{\partial t} = \beta \frac{\partial^2 n^\alpha}{\partial x^2}, \quad (35)$$

the same as in paper [6], where the diffusion coefficient

$$\beta = a_0^\alpha \frac{(n_0^\alpha)^2}{K^\alpha}, \quad (36)$$

out of which we can conclude that this equation is parabolic.

The fact that (34) is parabolic suggests the possibility that certain classes of sedimentation problems could be solved.

A brief illustration of the theory for the particle concentration and the diffusion coefficient, using (34) and (35), yields.

Figure 2 shows the measured particle concentration as a function of height in to the tube after 20,000 s of elapsed time, using values from literature of $a_0^\alpha = 15 \text{ kg/m}^2$, $K^\alpha = 3.1 \cdot 10^7 \text{ kg/m}^3\text{s}$, and the diffusion coefficient β in boundaries from 10^{-7} (data 1) to $10^{-10} \text{ m}^2/\text{s}$ (data 4).

Figure 3 shows the diffusion coefficient as a function of the local particle concentration using values of a_0^α and K^α in (35).

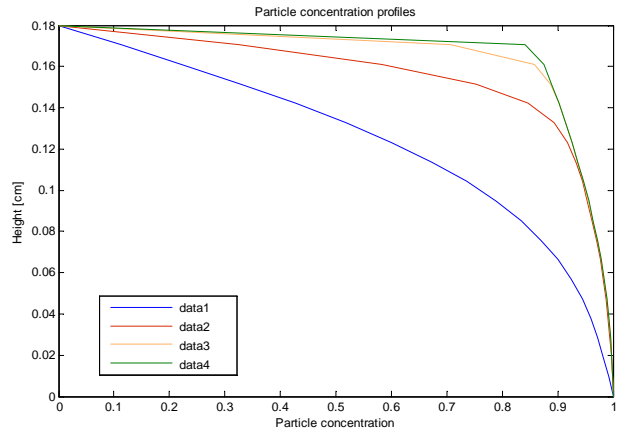


Figure 2. Particle concentration profiles

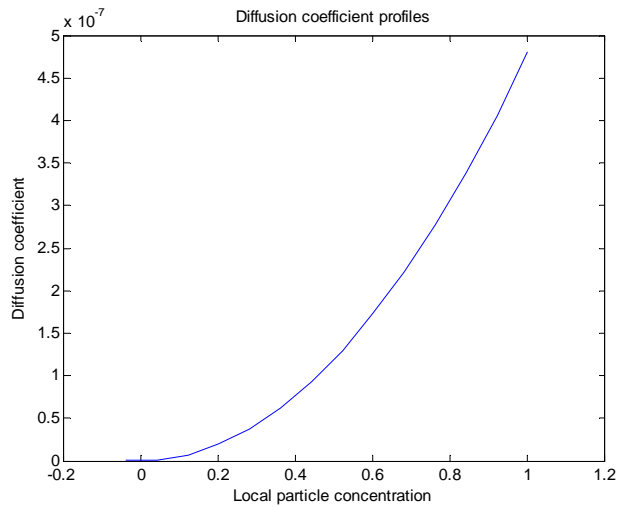


Figure 3. Diffusion particle coefficient

6. CONCLUSION

In this paper, the continuum theory has been applied to the case of two-phase (suspension) flow. The suspension is defined as a mixture of the basic phase (fluid) and the disperse phase (rigid particles). The basic equations of motion of different phases are derived as the model of one-dimensional sedimentation of rigid particles through the fluid in the direction of vertical axis. After using the basic equations of motion, the diffusion equation of quasi-steady sedimentation is derived and then graphically presented. The analysis of the obtained results shows that the suspension concentration distribution of rigid particles depends upon the diffusion coefficients. The diffusion equation is the same as in the paper by Hill and Bedford but derived using Hamilton's external variation principle.

REFERENCES

- [1] Bedford, A. and Hill, C.D.: A mixture theory for particulate sedimentation with diffusivity, *AICHE Journal*, Vol. 23, No. 3, pp. 403-404, 1977.
- [2] Drew, D. and Segel, L.: Averaged equations for two-phase flows, *Studies in Applied Mathematics*, Vol. 1, No. 2, pp. 205-231, 1971.
- [3] Ahmadi, G.: A generalized continuum theory for multiphase suspension flows, *International Journal*

- of Engineering Science, Vol. 23, No. 1, pp. 1-25, 1985.
- [4] Drew, D.: Mathematical modeling of two-phase flow, Annual Review of Fluid Mechanics, Vol. 15, No. 3, pp. 261-291, 1983.
- [5] Hill, C.D. and Bedford, A.: Stability of the equations for particulate sedimentation, The Physics of Fluid, Vol. 22, No. 2, pp. 1252-1254, 1979.
- [6] Hill, C.D., Bedford, A. and Drumheller, D.S.: An application of mixture theory to particulate sedimentation, Journal of Applied Mechanics, Vol. 47, No. 2, pp. 261-265, 1980.
- [7] Cvetković, P.: The continuum theory applied of suspension flow, in: *Proceedings of the XVII Yugoslav Meeting of Theoretical and Applied Mechanics*, 02-09.06.1986, Zadar, Croatia, pp. 125-128.
- [8] Bedford, A. and Drumheller, D.S.: Theories of immiscible and structured mixtures, International Journal of Engineering Science, Vol. 21, No. 8, pp. 863-960, 1983.
- [9] Jarić, J., Cvetković, P., Golubović, Z. and Kuzmanović, D.: *Advances in Continuum Mechanics*, Monographycal booklets in applied & computer mathematics, MV-25/PAMM, Budapest, 2002.
- [10] Hutter, K. and Schneider, Lukas: Important aspects in the formulation of solid-fluid debris-flow models. Part II. Constitutive modelling, Continuum Mechanics and Thermodynamics, Vol. 22, No. 5, pp. 391-411, 2010.

**О ПРИМЕНИ ТЕОРИЈЕ КОНТИНУУМА НА
СЛУЧАЈ ЈЕДНОДИМЕНЗИЈСКЕ
СЕДИМЕНТАЦИЈЕ**

**Бошко П. Цветковић, Драгослав С. Кузмановић,
Предраг А. Цветковић**

У раду примењена је теорија континуума на кретање суспензија. За дисперзну фазу узети су састојци мешавине, који су расподељени у основној фази, флуиду. Користећи основне законе баланса механике континуума и Лагранжеве множице, изведене су конститутивне једначине засићене суспензије. Добијене конститутивне једначине се користе за извођење опште једначине кретања основне и α -ог састојка дисперзне фазе у правцу вертикалне осе. Користећи опште једначине кретања и одговарајућа ограничења, изводи се једначина дифузије за случај квазистатичке седиментације која се и графички представља.