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On the dynamic modelling of flexible manipulators

Some of the issues involved in the dynamic modelling of single-link flexible manip- ulators are examined. In particular, we revisit the problem of reference frames, closed-form expressions of characteristic equation, mode shapes and orthogonality conditions when the assumed modes method is the method-of-choice for finite dimensionalization of the system, and the concept of geometric stiffening.

Keywords: *flexural-torsional coupling, payload, flexible manipulator, tip-mass, in-plane and out-of-plane deformation.*

1. INTRODUCTION

The manipulators of interest are those that can be idealized as beams and attached to a rotating or non-rotating hub. These manipulators are mainly observed in robotic applications where they are used to transfer parts or objects from one point to another. They are also widely used in automotive and aerospace industries for activities such as spray painting and welding. Traditionally, these manipulators are usually rigid and heavy.

However, the need for improved power consumption and efficiency has motivated the use of modern materials and manufacturing methods to construct flexible and lightweight manipulators. The attendant problems have been the increased complexity in the system dynamics and control. A recent survey on the subject of dynamics and control of flexible manipulators has been presented by Dwivedy and Eberhard [7].

Flexible manipulators are either single- or multi-link. A more detailed model, especially in the multi-link scenario, includes the modelling of the motors and joints as demonstrated in Refs. [3] and [11]. Each link is modelled using either Euler-Bernoulli beam theory or Timoshenko beam theory. The single-link model is presented in this paper.

The development of the system governing equations is usually based on Newton-Euler method or the energy methods of Lagrange or Hamilton's principle. Given that each link is a continuum, the problem is simplified by a finite dimensionalization process which is often approached via the assumed mode method or finite element method. In the former, the field variable is expanded as the sum of the products of eigenfunctions and undetermined parameters. The most commonly used eigenfunctions are those that relate to the non-rotating system. There are instances where the lumped parameter models are implemented.

We examine three issues and organize the paper accordingly. The first is the question of reference frame(s) selection to describe the system dynamics. Here we discuss the classical clamped (also called pseudo-clamped) and the nonclassical pseudo-pinned and pseudo- pinned-pinned reference frames.

The second problem of interest is the determination of the characteristics function of a given system, its eigenfunctions and orthogonality conditions. There are many reasons to seek closed-form expressions of the characteristics function and eigenfunctions. Apart from the potential to provide insights into the influence of some design parameters, closed-form expressions lead to smaller vector space when compared to finite element dimensionalization, a highly desirous characteristic in control implementation. In presenting the closed-form expressions, we examine a flexible manipulator with a tip load whose centre of mass is different from the point of attachment to the beam. A reference frame system is selected to highlight the role of the various components of the offset.

Finally, we revisit the problem of geometric stiffening. This is a motion induced effect that captures the role of centrifugal forces. Models that ignore geometric stiffening erroneously predict instability and allow the beam to rotate at its critical speed.

2. REFERENCE FRAME

Two types of reference frames are used in the analysis of a rotating flexible beam, namely, an inertial (or Newtonian) frame and a body-fixed (or non-inertial or rotating) frame. The consequence of the choice of the rotating frames may be anticipated to be limited to the form (i.e., simplicity or complexity) of the resulting system governing equations. Agrawal and Shabana [1] have highlighted the importance of the choice of the rotating frame. The most popular non-inertial frame in application is the clamped-free frame where the rotating abscissa (or x-axis) is tangent to the flexible beam at the base. This corresponds to the pseudo-clamped frame of Bellezza et al. [4]. They also examine the dynamics with a pseudo-pinned frame where the rotating abscissa passes through the centre of mass of the entire system. Their analysis was limited to Euler-Bernoulli beams and White and Heppler [31] extended it to Timoshenko beams. A schematic of these frames is illustrated in Fig.1.

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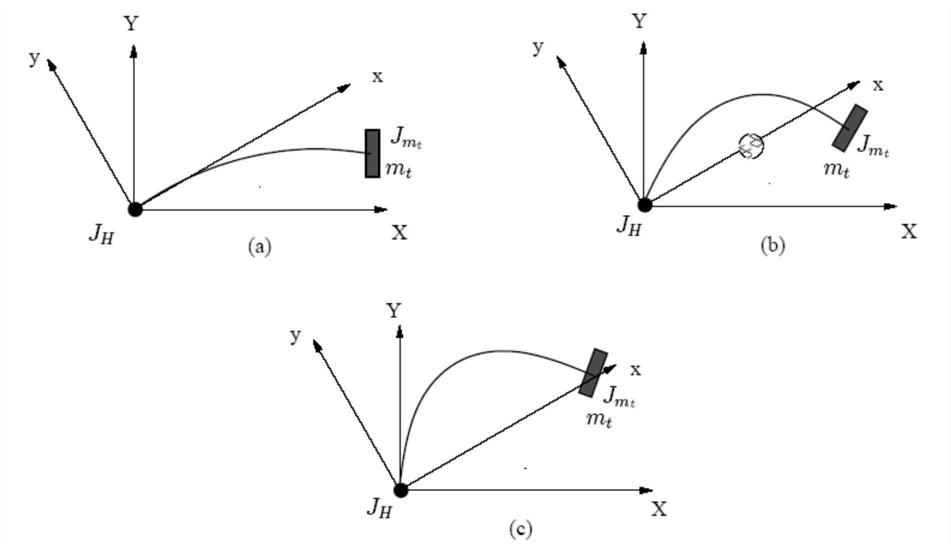

Figure 1. Frame of reference: (a) pseudo-clamped, (b) pseudo-pinned, and (c) pseudo-pinned-pinned

In the figure, the pseudo-pinned-pinned non-inertial frame where the rotating abscissa passes through the both ends of the beam was introduced by Oguamanam et al. [22]. In developing a finite-dimensional model, they showed that the modal co-ordinates obtained for the various non-inertial frames are related via transformation matrices. This therefore implies that the inertia (or mass) and stiffness matrices, and the force vector of the resulting equations of motion are transformable. In particular, if we denote the vector of modal co- ordinates of the pseudo-clamped, pseudo-pinned, and psuedo-pinned-pinned scenarios as qc , qp and qpp , respectively, it was shown that

$$q_p = T_{pp/p}\, q_{pp} \text{ and } q_c = T_{c/p}\, q_p , \qquad (1)$$

where the transformation matrix $T_{i/j}$ for $i, j \in \{c, p, p\}$ maps the j generalized modal co-ordinates to the i generalized modal co-ordinates.

3. CHARACTERISTICS EQUATION, MODE SHAPES AND ORTHOGONALITY CONDITION

The two competing methods for analyzing the dynamics of a rotating flexible beam are the use of assumed modes or finite element techniques. The former is rather sensitive to the selected modes and to the form in which they are expressed, while the latter suffers from the need for many degrees of freedom in order to improve the fidelity of the model. One way to improve the results with the assumed modes method is to use the modes from a corresponding non-rotating structure wherever possible. Further, it is desirous to obtain their closed-form expressions in conjunction with that of thesystem's characteristic (or frequency) equation because of the computational advantage they offer and the potential to quickly reveal the interdependencies of design variables or parameters.

The closed-form characteristics equation and mode shapes of uniform cross section flexible beams (modelled by the use of Euler-Bernoulli beam theory) in planar motion for various boundary conditions and without attachments are readily available in standard vibration textbooks such as that by Inman [10]. Some degree of complexity can be introduced by allowing attachments, non-uniform cross section, out-of-plane motion, and rotary inertia and or shear deformation. Some representative studies include Refs. [6], [8], and [14].

Low in [16] and [17] and Nayfeh et al. [18] provide various extensions by including the effects of base excitation and also hub inertia. A common theme inthese studies is that motion is planar and the centre of gravity of the tip load is coincident with the point of attachment to the beam. Attempts to address the latter issue can be found in the investigations by Bhat and Wagner [5], White and Heppler [30], and Kirk and Wiedemann [13].We find [5] to be simplistic and while the study by White and Heppler [30] is comprehensive, their approach did not explicitly permit the examination of the influence of each offset component. Kirk and Wiedemann [13], on the other hand, neither provided closed-form expressions nor included torsional effects.

The work by Oguamanam [19] in which planar elastic bending deformation and elastic torsional deformation are allowed is discussed in the following. The technique used to model the non-coincidence of the centre of gravity of the tip load and the point of attachment to the beam allowed for the examination of the influence of each offset coordinate or their combination for a given payload mass and inertia tensor about the centre of gravity.A schematic of the flexible beam with a rigid tip load is depicted in Fig. 2. The length of the beam is denoted by L, A denotes the cross-sectional area, I denotes the second moment area about the bending axis, and the polar moment of inertia is denoted by J. The density, Young's modulus, and shear modulus of the material from which the beam is constructed are respectively denoted by ρ, E, and G. The mass of the rigid tip load is denoted by m_t and its inertia tensor about the centre of gravity is represented by I.

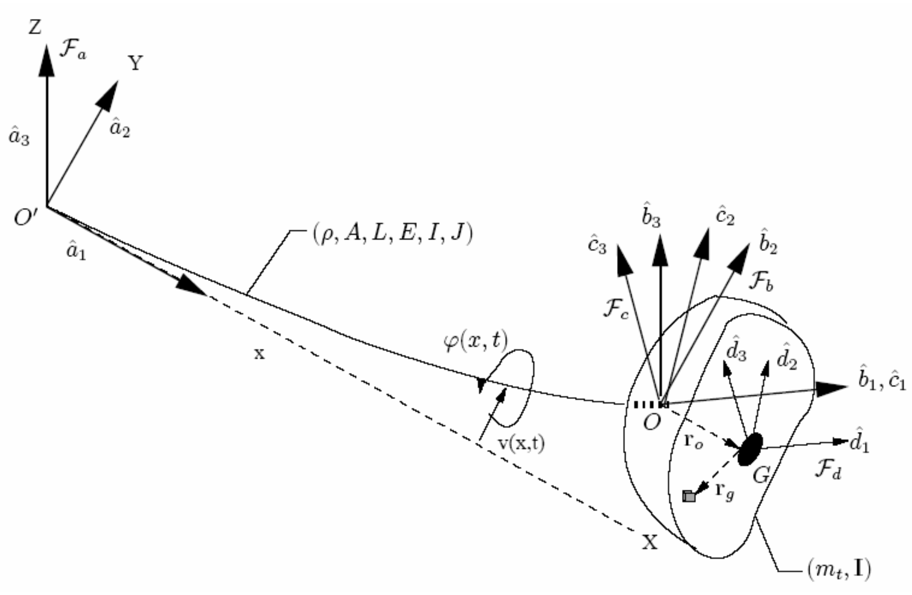

Figure 2. Cantilevered beam with a tip mass whose centre of gravity is not coincident with the point of attachment

The deformation of the system from its original configuration is characterized by the use of four orthogonal dextral references frames, viz. $\mathcal{F}_a$, $\mathcal{F}_b$, $\mathcal{F}_c$, and $\mathcal{F}_d$. The reference frame $\mathcal{F}_a$ is an inertia (or a Newtonian) frame which has its origin fixed to the clamped end O\` of the flexible beam. The frame is identified by X-,Y-, and Z-coordinate axes with the corresponding unit vectors respectively denoted by $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$. The X-axis coincides with the neutral axis of the beam before deformation. The payload is attached to the tip of the beam at point O. A dextral beam body-fixed reference frame $\mathcal{F}_b$ with unit vectors b_1, b_2, and b_3 is affixed at this point of attachment such that each unit vector b_i is parallel to the corresponding $\hat{a}_i$ before deformation. The b_3 axis is always parallel to the $\hat{a}_3$ axis. The penultimate dextral body-fixed reference frame $\mathcal{F}_c$ with unit vectors $\hat{c}_1$, $\hat{c}_2$, and $\hat{c}_3$, is affixed at the point of attachment of the payload with each $\hat{c}_i$ parallel to the corresponding b_i. This frame is fixed to the payload and the arrangement is such that $\hat{c}_1$ and b_1 are always parallel.Finally, a dextral payload body-fixed reference frame $\mathcal{F}_d$, with unit vectors d_1, d_2, and d_3, is attached to the centre of gravity of the payload such that its unit vectors are always parallel to the corresponding unit vectors of reference frame Fc. The position vector from the point of attachment of the payload O to the centre of gravity of the payload G is denoted as r_0 and it has the Cartesian components o_x, o_y, and o_z measured in the $\mathcal{F}_c$ frame. A differential beam element located at position x from the clamped end of the beam is assumed to experience both torsional deformation $\varphi(x,t)$ and planar (XY-plane) bending deformation $v(x,t)$. The system governing equations are obtained by using energy principles and the Hamilton's principle. The set of governing equations of motion are uncoupled, but the flexural and torsional displacement couple through the boundary conditions. Further a variable transformation is necessary if the separable method is to be used in solving the system of equations. This is because the boundary conditions are nonhomogeneous. To this end, a new variable γ is introduced and defined as

$$\gamma(x,t) = \varphi + \frac{m_t g}{GJ} O_y x \, . \tag{2}$$

We assume separable solutions are assumed as

$$v(x,t) = LV(\xi)e^{i\omega t} \quad \text{and} \quad \gamma(x,t) = \Gamma(\xi)e^{i\omega t} \, , \tag{3}$$

where ξ is a nondimensional position along the span of the beam, $V(\xi)$ and $\Gamma(\xi)$ are mode shapes, and ω denotes the radial frequency. If use is made of the geometric boundary conditions resulting from the clamped left end, it can be shown that the general solution to the governing equations can be written respectively as

$$V(\xi) = A_1\left(\sin\left(\lambda\xi\right) - \sinh\left(\lambda\xi\right)\right) + \\ + A_2\left(\cos\left(\lambda\xi\right) - \cosh\left(\lambda\xi\right)\right) , \tag{4}$$

$$\Gamma\left(\xi\right) = B\sin\left(\lambda^2 \chi\mu\xi\right) . \tag{5}$$

These equations are now substituted into the remaining boundary conditions to obtain a set of equations which can be written in matrix notation as

$$\mathrm{A}_{3\times3}\mathrm{x}_{3\times1} = 0 \, . \tag{6}$$

where x = [B A_1 A_2 $]^T$ is the column vector of the coefficients of the general solutions, Eqs.(4) and (5).

3.1. The frequency equation

The frequency equation is obtained by equating to zero the determinant of matrix **A** in Eqn. (6), and it may be written as

$$
\begin{aligned}
&\mu c_{1u}\mathrm{F}_{cf} - \lambda^2\chi\mathcal{I}_{xxt}s_{1u}\mathrm{F}_{cf} - \lambda^3\mu\mathcal{I}_{zzt}c_{1u}\mathrm{F}_{cr} + \\
&\lambda^5\chi s_{1u}\left(\mathcal{I}_{xxt}\mathcal{I}_{zzt} - \mathcal{I}_{xzt}^2\right)\mathrm{F}_{cr} - M_t\left[\lambda\mu c_{1u}\,\mathrm{F}_{cs} + \right. \\
&2\lambda^2\mu a_x c_{1u}s_1 sh_1 - \lambda^3\chi\mathcal{I}_{xxt}s_{1u}\mathrm{F}_{cs} - \lambda^4\mu\mathcal{I}_{zzt}c_{1u}\mathrm{F}_{cc} + \\
&2\lambda^4\chi s_{1u}\left(a_z\mathcal{I}_{xzt} - a_x\mathcal{I}_{xxt}\right)s_1 sh_1 + \lambda^6\chi s_{1u}\left(\mathcal{I}_{xxt}\mathcal{I}_{zzt} - \right. \\
&\left.\left.\mathcal{I}_{xzt}^2\right)\mathrm{F}_{cc}\right] - M_t^2\left[\lambda^3\chi a_z^2 s_{1u}\mathrm{F}_{cs} + \lambda^4\mu a_x^2 c_{1u}\mathrm{F}_{cc} - \right. \\
&\left.\lambda^6\chi s_{1u}\left(a_x^2\,\mathcal{I}_{xxt} - 2a_x a_z\mathcal{I}_{xzt} + a_z^2\,\mathcal{I}_{zzt}\right)\mathrm{F}_{cc}\right] = 0,
\end{aligned}
\tag{7}
$$

where

$$
\begin{gathered}
\mathrm{F}_{cf} = 1 + c_1 ch_1,\quad \mathrm{F}_{cr} = s_1 ch_1 + c_1 sh_1, \\
\mathrm{F}_{cs} = s_1 ch_1 - c_1 sh_1,\ \mathrm{F}_{cc} = 1 - c_1 ch_1, \\
M_t \overset{\Delta}{=} \frac{m_t}{\rho AL},\quad \lambda^4 = \frac{\rho AL^4\omega^2}{EI}, \\
s_{1u} = \sin\left(\lambda^2\chi\mu\right),\quad c_{1u} = \cos\left(\lambda^2\chi\mu\right), \\
s_1 = \sin(\lambda),\quad \chi^2 = \frac{EI}{GJ},\quad \mu^2 = \frac{J}{AL^2}, \\
c_1 = \cos(\lambda),\quad sh_1 = \sinh(\lambda), ch_1 = \cosh(\lambda), \\
a_i \overset{\Delta}{=} \frac{a_i}{L}\quad i\in\{x,y,z\},\quad \mathcal{I}_{ij} \overset{\Delta}{=} \frac{I_{ij}}{\rho AL^3}\quad i,j\in\{x,y,z\} \\
\mathcal{I}_{xxt} = \mathcal{I}_{xx} + M_t\left(a_y^2 + a_z^2\right), \\
\mathcal{I}_{zzt} = \mathcal{I}_{zz} + M_t\left(a_x^2 + a_y^2\right), \\
\mathcal{I}_{xzt} = \mathcal{I}_{xz} + M_t a_x a_z.
\end{gathered}
\tag{8}
$$

The effect of varying the moment of inertia of the tip mass about the z- and x-axis is illustrated in Fig. 3. We note that the sensitivity of the natural frequency to I_{zz} appears to be dependent on the relative magnitude of I_{xx}. We also note, for the specific example, the presence of a long range of I_{xx} for a given I_{zz} for which the natural frequency is unaffected before a major drop in magnitude is observed. These effects are perhaps attributable to the increasing influence of torsional deformation.

3.2. The mode shapes

We ignore the bending moment boundary condition at the free end (or tip) of the beam and derive the coefficients A_1 and B of the mode shapes in terms of A_2. These coefficients, for brevity, are redefined as

$$
B \overset{\Delta}{=} \frac{B_{numer}}{B_{denom}}A_2 \ \text{ and } \ A_1 \overset{\Delta}{=} \frac{A_{1_{numer}}}{A_{1_{denom}}}A_2,
$$

where B_{numer} is

$$
\begin{aligned}
&4\chi I_{xz}\lambda^3\left(s_1 ch_1 + c_1 sh_1\right) + \\
&+4\chi\lambda^2 M_t\left(a_x a_z\lambda\left(s_1 ch_1 + c_1 sh_1\right) + \right. \\
&\left.+a_z s_1 sh_1 - \lambda^2 I_{xz}\left(1 - c_1\, ch_1\right)\right),
\end{aligned}
\tag{9}
$$

the A_{1numer} term represents

$$
\begin{aligned}
&2\left(\mu c_{1u} - 2\chi I_{xx}\lambda^2 s_{1u}\right)\left(sh_1 - s_1\right) + \\
&2M_t\left(\chi\left(a_y^2 + a_z^2\right)s_{1u}\left(s_1 - sh_1\right) + \right. \\
&\lambda^2\left(\chi\lambda^2\left(a_z I_{xz} - a_x I_{xx}\right)s_{1u} + \mu a_x c_{1u}\right)\left(s_1 - sh_1\right) + \\
&\left.\lambda\left(\chi\lambda^2 I_{xx}s_{1u} - \mu c_{1u}\right)\left(c_1 - ch_1\right)\right) + \\
&2\chi a_y^2\lambda^3 M_t^2\left(\left(c_1 - ch_1 - a_x\lambda\left(s_1 - sh_1\right)\right)\right)s_{1u}.
\end{aligned}
\tag{10}
$$

and the terms B_{denom} and A_{1denom} are equivalent and may be written as

$$
\begin{aligned}
&-2\left(\mu c - \chi I\lambda s\right)\left(c - ch\right) + \\
&2M\left(\chi\left(a + a\right)\lambda s\left(c - ch\right) + \right. \\
&\lambda\left(\mu c - \chi I\lambda s\right)\left(s - sh\right) + \\
&\left.\lambda\left(a\left(\mu c - \chi I\lambda s\right) + a\chi I\lambda s\right)\left(c - ch\right)\right) + \\
&2\chi a\lambda M s\left(sh - s + a\lambda\left(ch - c\right)\right).
\end{aligned}
\tag{11}
$$

3.3 The orthogonality condition

The orthogonality condition is derived by using the system governing equations and the boundary conditions. Following the standard technique, the resulting expression may be written as

$$
\begin{aligned}
&\int\left(VV + \mu\Gamma\Gamma\right)\mathrm{d}\xi + MV(1)V(1) + \\
&I\Gamma(1)\Gamma(1) + Ma\left(V'(1)V(1) + V'(1)V(1)\right) + \\
&IV'(1)V'(1) + Ma\left(V(1)\Gamma(1) + V(1)\Gamma(1)\right) + \\
&I\left(V'(1)\Gamma(1) + V(1)\Gamma(1)\right) = \delta.
\end{aligned}
\tag{12}
$$

We close this section by mentioning extensions to the study in Refs. [9], [20], and [25].

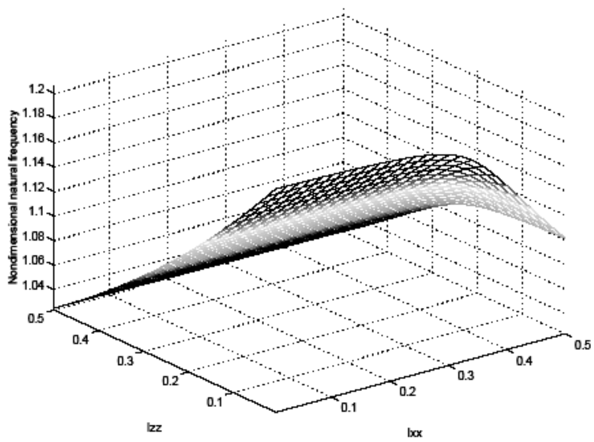

Figure 3. Effect of varying the moment of inertia of the mass about the z- and x-axes (for I_{xz}=0.005, a_x=a_y=a_z=0.05, Mt=1.0)

4. GEOMETRIC STIFFENING

In 1987, Kane at al. [12] brought to fore the potential inaccuracies in the results of most commercial software that were dedicated to the dynamics of multibody flexible systems. The problem was the errors in those formulations that inadvertently ignore the stiffening effect the rotating speed of a rotating flexible beam has on the beam. This effect is also generally identified by, amongst other terms, centrifugal stiffening, fortshortening, geometric stiffening, gyroscopic, or stress stiffening. The argument against Ref. [12] is the method for arriving at the formulation. This issue is addressed by Simo and Vu-Quoc [28] who presented a systematic method for handling non-linearities from which stiffening effects can be observed. An examination of the various methods to model stiffening effect is reported in Refs. [2] and [26], while Piedbœuf and Moore [23] investigated the role of the various techniques in the derivation of the governing equations of dynamic systems for both continuous and discrete models.Generally, the physics of a rotating beam indicates that it would stiffen with increasing rotating speed. However, models that assume only transverse deformation cannot capture this stiffening effect. The consequence is that the resulting model predicts a softening effect during which the beam can be rotated at a critical speed, i.e., the first natural frequency of the beam, before it becomes unstable. The issue of whether to include or exclude stiffening effects in a dynamic analysis is rather subjective. It is acknowledged that the severity of stiffening effect is dependent upon rotating speed, but a logical method to determine the threshold speed is still an issue of research. Mostly, it is assumed that the rotating speed is not sufficiently high to justify the added computation. Ryu et al. [24] have, however, postulated a method that is based on the comparison of frequencies.We illustrate the role of stiffening via a formulation that is based on the pseudo-clamped frame as depicted in Fig. 1a. The rotating beam is modelled using the Timoshenko beam theory. The transverse displacement and and cross-sectional rotation are denoted by ω(*x*, *t*) and ψ(*x*, *t*), respectively.Centrifugal forces are incorporated via their specific work done, $\mathrm{d}W_{cf}$ which is given as [21]

$$\mathrm{d}W_{cf} = -F_{cf}\left(\mathrm{d}s - \mathrm{d}x\right) \approx -\left\{\left(m_t L + \frac{1}{2}\rho A\left(L^2 - x^2\right)\right)\right\}\frac{1}{2}\left(\frac{\partial \omega}{\partial x}\right)^2 \mathrm{d}x. \tag{13}$$

The kinetic and potential energies of the system are derived. These and Eq. 13 are substituted into the extended Hamilton's principle which is then discretized using the expanded forms of the field variables such that

$$\omega\left(x,t\right) = \mathrm{W}^T(x)\mathrm{p}(t) \text{ and } \psi\left(x,t\right) = \Psi^T(x)\mathrm{q}(t)\,, \tag{14}$$

where W and Ψ are column vectors of basis functions and p and q are column vectors of undetermined parameters. The resulting finite dimensional equations of motion are written in matrix format as

$$\begin{aligned}
&\begin{bmatrix} \mu_{11} & m_{12} & m_{13} \\ m_{12}^T & \mathrm{M}_{22} & 0 \\ m_{13}^T & 0 & \mathrm{M}_{33} \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{\mathrm{p}} \\ \ddot{\mathrm{q}} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{22} & k_{23} \\ 0 & k_{33}^T & k_{33} \end{bmatrix}\begin{Bmatrix} \theta \\ \mathrm{p} \\ \mathrm{q} \end{Bmatrix} - \\
&\dot{\theta}^2\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Xi_{22} & 0 \\ 0 & 0 & \Xi_{33} \end{bmatrix}\begin{Bmatrix} \theta \\ \mathrm{p} \\ \mathrm{q} \end{Bmatrix} + \ddot{\theta}c_{m_t}\begin{bmatrix} 0 & c_{12}^T & c_{13}^T \\ 0 & 0 & \frac{1}{2}c_{23}^T \\ 0 & -\frac{1}{2}c_{23}^T & 0 \end{bmatrix}\begin{Bmatrix} \theta \\ \mathrm{p} \\ \mathrm{q} \end{Bmatrix} + \\
&\ddot{\theta}c_{m_t}\begin{bmatrix} 0 & c_{12}^T & c_{13}^T \\ -\frac{1}{2}c_{12} & 0 & c_{23}^T \\ -\frac{1}{2}c_{13}^T & -c_{23} & 0 \end{bmatrix}\begin{Bmatrix} \dot{\theta} \\ \dot{\mathrm{p}} \\ \dot{\mathrm{q}} \end{Bmatrix} + \\
&c_{m_t}\begin{bmatrix} \begin{bmatrix} \ddot{\mathrm{p}}^\mathrm{T} & \ddot{\mathrm{q}}^\mathrm{T} \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2}c_{23}^T \\ -\frac{1}{2}c_{23}^T & 0 \end{bmatrix}\begin{Bmatrix} \mathrm{p} \\ \mathrm{q} \end{Bmatrix} \\ 0 \\ 0 \end{bmatrix} + \\
&\ddot{\theta}\begin{bmatrix} \begin{bmatrix} \mathrm{p}^\mathrm{T} & \mathrm{q}^\mathrm{T} \end{bmatrix}\begin{bmatrix} \Xi_{22} & 0 \\ 0 & \Xi_{33} \end{bmatrix}\begin{Bmatrix} \mathrm{p} \\ \mathrm{q} \end{Bmatrix} \\ 0 \\ 0 \end{bmatrix} + \\
&\ddot{\theta}\begin{bmatrix} \begin{bmatrix} \dot{\mathrm{p}}^\mathrm{T} & \dot{\mathrm{q}}^\mathrm{T} \end{bmatrix}\begin{bmatrix} \Xi_{22} & 0 \\ 0 & \Xi_{33} \end{bmatrix}\begin{Bmatrix} \mathrm{p} \\ \mathrm{q} \end{Bmatrix} \\ 0 \\ 0 \end{bmatrix} = \begin{Bmatrix} c \\ 0 \\ c_{\psi_0} \end{Bmatrix}.
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
&\mu_{11} = \frac{\rho A L^3}{3} + \rho I L + J_H + J_{m_t} + m_t L^2, \\
&m_{12} = \int_0^L \rho A x \mathrm{W}^T dx + m_t L \mathrm{W}_L^T\,, \\
&m_{13} = \int_0^L \rho I \Psi^T dx + J_H \Psi_0^T + J_{m_t}\Psi_L^T\,, \\
&m_{22} = \int_0^L \rho A \mathrm{W}\mathrm{W}^T dx + m_t \mathrm{W}_L \mathrm{W}_L^T\,, \\
&m_{33} = \int_0^L \rho I \Psi\Psi^T dx + J_H \Psi_0 \Psi_0^T + J_{m_t}\Psi_L\Psi_L^T\,, \\
&\Xi_{22} = \int_0^L \rho A \mathrm{W}\mathrm{W}^T dx + m_t \mathrm{W}_L \mathrm{W}_L^T - \int_0^L \vartheta \mathrm{W}'\mathrm{W}'^T dx, \\
&\Xi_{33} = \int_0^L \rho I \Psi\Psi^T dx + J_{m_t}\Psi_L\Psi_L^T\,, \\
&c_{12} = 2\mathrm{W}_L^T \quad c_{13} = -2L\Psi_L^T \quad c_{23} = -2\mathrm{W}_L\Psi_L^T\,, \\
&k_{22} = \int_0^L \kappa G A \mathrm{W}'\mathrm{W}'^T dx \quad k_{23} = -\int_0^L \kappa G A \mathrm{W}'\Psi'^T dx, \\
&k_{33} = \int_0^L \kappa G A \Psi\Psi^T dx + \int_0^L E I \Psi'\Psi'^T dx, \\
&\vartheta = m_t L + \frac{1}{2}\rho A\left(L^2 - x^2\right).
\end{aligned} \tag{16}$$

The above equations of motion presupposes that the velocity (or rotating speed) profile is not prescribed a priori. If this were not the case, the appropriate equations of motion can be easily deduced by deleting the first row and moving the first column to the right-hand side. The corresponding equations of motion for the case of Euler-Bernoulli beam theory are obtained by taking the limit $\kappa G \to \infty$ which yields zero shear strain and $\psi = \partial\omega / \partial x$. Fig. 4 depicts the tip-displacement profiles obtained for both linear and nonlinear models using Timoshenko beam model. The corresponding results for Euler-Bernoulli beam model are illustrated in Fig. 5. These results are for the scenarios in which the velocity (or rotating speed) profile is specified. The first three modes have been retained in the analysis. We observe that waveform of the linear models have smaller maximum magnitudes and smaller frequency than the corresponding nonlinear models. The comparison of the two beam theories shows that the predicted deflection by the Timoshenko beam is higher than that by the Euler-Bernoulli beam theory.The investigations by Liu and Hong [15] and by Trindade and Sampaio [29] are extensions to this study.

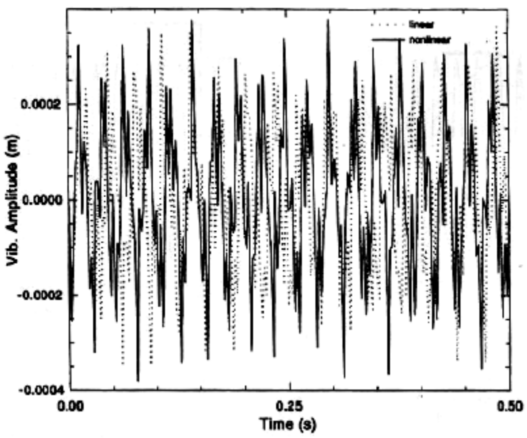

Figure 4. Timoshenko beam tip displacement ($\dot{\theta} = 50\ \mathrm{s}^{-1}$, $h/L = 0.01$)

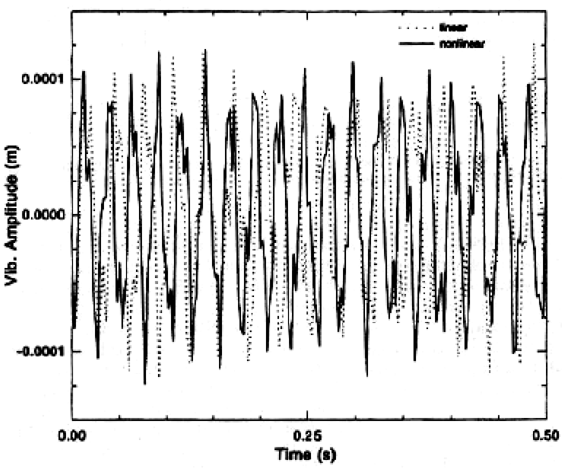

Figure 5. Euler-Bernoulli beam tip displacement ($\dot{\theta} = 50\ \mathrm{s}^{-1}$, $h/L = 0.01$)

5. CONCLUSION

Three basic issues that arise in the dynamic modelling of flexible manipulators have been discussed via the use of a single-link manipulator. These are the selection of reference frames, the determination of closed-form eigenfunctions expressions for use in the assumed mode method, and the role of geometric stiffening. It is observed that the choice of reference frames will affect the complexity of the resulting system equations of motion, but there are transformations that map one frame to the other. In determining the eigenfunctions, the role of tip-mass is treated in a more generalized form by using reference frames that permit a lucid reflection of the coordinates of the offset of the centre of gravity of the tip mass from the point of attachment to the beam.

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