Fragment Size Distribution in Dynamic Fragmentation: Geometric Probability Approach

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Dynamic fragmentation is a complex phenomenon inherent in numerous natural and engineering systems. Determination of the fragment size (or mass) distribution law is one of the most important objectives in dynamic fragmentation modeling. In the present paper, a general approach based on the simple assumption of random geometric partition of a body has been considered. Starting from the binomial distribution of fracture sites (points, lines or planes), size distribution laws are derived for 1D, 2D and 3D geometries. Geometric fragmentation models based on Mott's and Grady-Kipp's approaches are analyzed. The models originating from the Voronoi diagrams are also considered. The results of presented models are compared with numerical simulations and experimental data, showing significant compatibility as well as certain limitations. It has been concluded that preferred theoretical model depends on dimensionality of fragmentation process.

Keywords: *dynamic fragmentation, geometric probability, probability distribution, Voronoi diagrams, military applications.*

1. INTRODUCTION

Fragmentation is a process of disintegration of a body caused by multiple fractures of the material. It is a ubiquitous phenomenon in nature and engineering systems, which takes place in different size and time scales. Expansion of galaxies, asteroid impacts, explosively driven fragmentation, and fragmentation induced by impact of nuclei are the most important examples of dynamic fragmentation.

Military applications have been the main motivation for both experimental investigations and theoretical studies in the field of dynamic fragmentation. Studies of fragmentation of metallic rings and shells have been applied to the analysis and design of high-explosive fragmentation warheads. Complete characterization of a fragmentation process implies determination of fragment size, shape and mass distribution, fragment velocity distribution, as well as spatial distribution of generated fragments. We will here focus on the consideration of the fragment size distribution.

Fragmentation modeling is extremely difficult problem, involving complex physics dependent on loading conditions, material characteristics and problem geometry. There are several approaches to the fragmentation problem – empirical [1], probabilistic [2-4], energetic [5], approach based on fracture mechanics [6], numerical approach [7,8], etc.

In the present paper, fragmentation problem has been treated from the general approach based on geometric statistics.

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2. FRAGMENT SIZE DISTRIBUTION MODELING BY GEOMETRIC PROBABILITY

Let us consider dynamic fragmentation of a homogenous body brought about by impulsive loads. The simplest approach to the modeling of size distribution of generated fragments is based on the assumption of random distribution of “fracture sites”. Therefore, fragmentation process in 1D, 2D and 3D is considered to be equivalent to the random segmentation of a line, area and volume, respectively.

2.1 One-dimensional fragmentation

Let us first consider probabilistic fragmentation of a line produced by random selection of “fracture” sites, following the approach established by Linaeu [2] and Mott [1] and improved in [9] and [10]. Assuming n break points randomly distributed on the line of length L, the probability of finding exactly k ($k \le n$) “fracture” points on a line segment of length l ($l \le L$) is determined by binomial distribution

$$P_1(n,k) = \binom{n}{k} p_1^k (1-p_1)^{n-k}, \tag{1}$$

where $p_1 = l/L$ is the probability that a point will fall into the segment of length l. If we restrict consideration to the case of a large number of generated fragments, both the line length L and the number of fracture points n tend to infinity, and probability (1) transforms into the Poisson distribution of the form:

$$P(k,l) = e^{-\lambda l} \frac{(\lambda l)^k}{k!}, \tag{2}$$

where $\lambda = n/L$ is the number of points per unit length, reciprocal to the average fragment size $\overline{l}$. Now, using the Poisson distribution (2), the probability dp of finding fragment of a length in the interval $[l, l + \mathrm{d}l]$ can be calculated as follows. The probability that there is no point on the line segment of length l is

$$P(0,l) = \mathrm{e}^{-\lambda l}, \quad (3)$$

and that there is exactly one point in the adjacent segment of length $\mathrm{d}l$ is defined by

$$P(1,\mathrm{d}l) = \lambda \mathrm{d}l\,. \quad (4)$$

The probability $\mathrm{d}p$ can be determined as

$$\mathrm{d}p = P(0,l)P(1,\mathrm{d}l) = \lambda \mathrm{e}^{-\lambda l}\mathrm{d}l\,. \quad (5)$$

Therefore, the probability density function of fragment length is

$$f(l) = \frac{\mathrm{d}p}{\mathrm{d}l} = \lambda \mathrm{e}^{-\lambda l}\,. \quad (6)$$

The cumulative probability distribution function $P(>l) = P(l)$ (that the fragment length is greater than l) has the exponential form

$$P(l) = \int_l^\infty f(l)dl = \mathrm{e}^{-\lambda l}\,. \quad (7)$$

The Lineau (exponential) distribution (7) successfully describes the fragment size distribution in different fragmentation events [9]. It is also the framework for several advanced fragmentation models.

Similar statistical approach to the random geometric fragmentation is proposed in [4]. However, this model uses an important additional assumption: existence of the minimum fragment size $l_{\min}$. The experimental studies [11,12], as well as energy based theoretical model [5] strongly justify this assumption. The minimum fragment size is the consequence of unloading waves produced at the separation points. The regions of material traversed by these waves are prevented from further failure [3,13]. The analysis of expected number of fragments [4] provides the relation between the average and minimum fragment length

$$\frac{\bar{l}}{l_{\min}} = 3.25\,. \quad (8)$$

However, numerical approach to the 1D fragmentation based on cohesive model of crack behavior [8,14] yields the modification of this relation

$$\frac{\bar{l}}{l_{\min}} = 2.70\,. \quad (9)$$

Using maximum entropy principle [15], it can be shown that cumulative fragment size distribution law has the exponential form

$$P(l) = \mathrm{e}^{-\lambda_\mathrm{r}(l - l_{\min})},\ l \ge l_{\min} \quad (10)$$

where λ_r is a distribution parameter. The average fragment length is determined by

$$\bar{l} = \frac{1}{\lambda_r} + l_{\min}\,. \quad (11)$$

One should note that the fragment size distribution (10) is completely defined by the average fragment length, because parameter $l_{\min}$ is defined by (8) or (9) and λ_r can be calculated from (11). The distribution (10) that uses relation (8) will be referred to as Zhang distribution, whereas the same distribution law with relation (9) will be termed as modified Zhang distribution.

Another possible approach to the random geometric fragmentation is based on Voronoi-Dirichlet diagrams [16]. The Voronoi-Dirichlet decomposition of a space (1D, 2D or 3D) with randomly generated initial points imply partitioning of a space in a certain number of subspaces such that each subspace contains exactly one generating point and every point in a given subspace is closer to its generating point than to any other. Voronoi-Dirichlet algorithm may be used as a model for different physical processes, including fragmentation [17]. In 1D case, Voronoi segmentation (i.e. random fragmentation) of a line is defined by the midpoints of each pair of adjacent randomly distributed initial points. Using similar procedure as in the previous case [17,18], the probability density function of fragment length can be determined as

$$f(l) = 4\lambda^2 l \mathrm{e}^{-2\lambda l}, \quad (12)$$

and cumulative distribution reads

$$P(l) = (1 + 2\lambda l)\mathrm{e}^{-2\lambda l}\,. \quad (13)$$

Normalized ($\bar{l} = 1$) Lineau, Zhang, modified Zhang and Voronoi distribution are plotted in Figure 1.

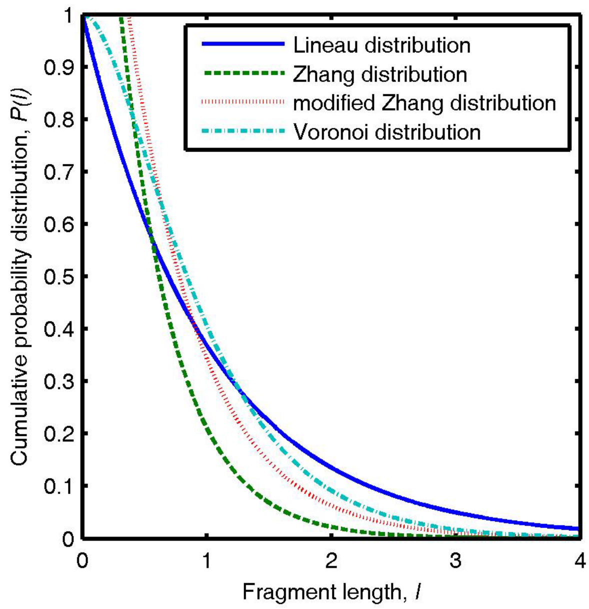

Figure 1. One-dimensional fragmentation: comparison of the Lineau, Zhang, modified Zhang and Voronoi model of fragment size distribution; the average fragment length is the same for each distribution

2.2 Two-dimensional fragmentation

Two-dimensional fragmentation is of much more practical importance, as it involves fragmentation of thin plates and shells. As a paradigmatic example, random fragmentation of a plane will be analyzed first.

The simplest method for plane fragmentation is by random generation of horizontal and vertical lines. If we assume that the two sets of lines are independent, with the same density λ, then Lineau distribution can be applied to both horizontal and vertical set of lines, yielding the cumulative area distribution

$$P(a) = \lambda^2 \iint\limits_{xy>a} \mathrm{e}^{-\lambda(x+y)} \mathrm{d}x\mathrm{d}y = 2\lambda\sqrt{a}\mathrm{K}_1(2\lambda\sqrt{a}) \tag{14}$$

where $\mathrm{K}_1(\bullet)$ is the modified Bessel function of the second kind of order 1 and a is a fragment area.

The analysis of fragments produced by detonation of high-explosive shells [1] suggested the fragment area distribution of the form:

$$P(a) = \mathrm{e}^{-\sqrt{\alpha a}}. \tag{15}$$

In the well-known Mott distribution law (15), the parameter α is related to the average fragment area by $\alpha = 2/\bar{a}$. Justification for this distribution is the fact that it is analogous to the Lineau exponential law (7), having in mind that for 2D fragmentation fragment length $l \sim a^{1/2}$.

Different postulate is offered in [19] and [9]: all fragment area distributions have the same probability, provided constant sum of fragments' area. This is equivalent to the Lineau 1D distribution, so the fragment area distribution law has the form

$$P(a) = \mathrm{e}^{-\alpha a}, \tag{16}$$

where α is reciprocal to the average fragment area. The fragment distribution law (16) will be referred to as Grady-Kipp model.

Finally, the approximate generalization of the fragment size distribution (13) generated by Voronoi diagrams is suggested [18]:

$$P(s) = \frac{\Gamma(n, \mu n s)}{\Gamma(n)}, \tag{17}$$

where $\Gamma(\bullet)$ and $\Gamma(\bullet,\bullet)$ are the complete and upper incomplete gamma function, s is fragment size (length, area or volume), μ is the reciprocal to the average fragment size, and n = 2, 4 and 6, corresponds to 1D, 2D and 3D Voronoi distribution, respectively. For 2D case (n = 4), the cumulative distribution (17) reads:

$$P(a) = \frac{\Gamma(4, 4\alpha a)}{\Gamma(4)} = \mathrm{e}^{-4\alpha a}\sum_{k=0}^{3}\frac{(4\alpha a)^k}{k!}. \tag{18}$$

The Bessel, Mott, Grady-Kipp and Voronoi 2D distributions are shown in Figure 2.

2.3 Three-dimensional fragmentation

Three-dimensional fragmentation implies fractures through all three dimensions of a fragmentation body. This is the most complex and the most important case of fragmentation from the application aspect. The main example is fragmentation of a space with three sets of parallel and mutually orthogonal planes. Supposing that the three sets of planes are independently distributed with the same density λ, the Lineau approach leads to the cumulative distribution:

$$P(v) = \lambda^3 \iiint\limits_{xyz>v} \mathrm{e}^{-\lambda(x+y+z)} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \lambda\sqrt{\lambda v}\ \mathrm{G}_{0,\,3}^{3,\,0}\left(\lambda^3 v \,\middle|\, \begin{matrix} - \\ -\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2} \end{matrix}\right) \tag{19}$$

where G is Meijer's G function [20].

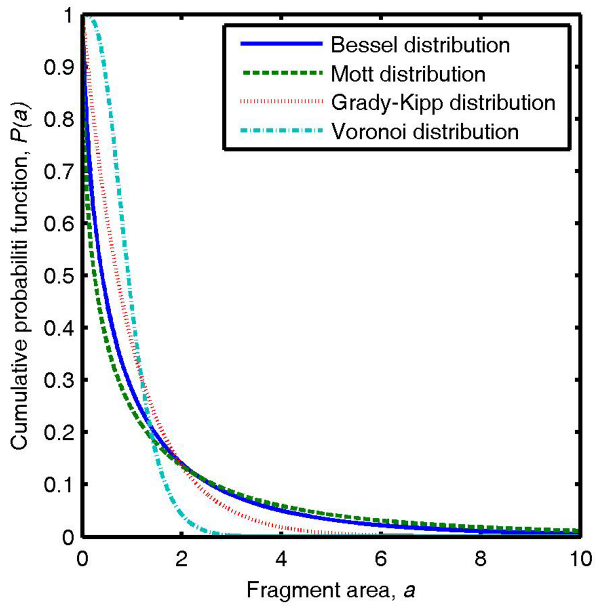

Figure 2. Two-dimensional fragmentation: comparison of the Bessel, Mott, Grady-Kipp and Voronoi model of fragment area distribution; the average fragment area is the same for each distribution

The Mott's formula for 3D case, having in mind that fragment length $l \sim v^{1/3}$, has the form

$$P(v) = \mathrm{e}^{-\sqrt[3]{\gamma v}}. \tag{20}$$

The parameter γ is defined by $\gamma = 6/\bar{v}$, where $\bar{v}$ is the average fragment volume.

Following the same argument as in (16), Grady-Kipp's cumulative fragment distribution can be written in the form:

$$P(v) = \mathrm{e}^{-\gamma v}, \tag{21}$$

where γ is reciprocal to the average fragment volume.

Applying n = 6 to (17), the Voronoi distribution for 3D case becomes:

$$P(v) = \Gamma(6, 6\gamma v)/\Gamma(6) = \mathrm{e}^{-6\gamma v}\sum_{k=0}^{5}\frac{(6\gamma v)^k}{k!}. \tag{22}$$

Four analyzed cumulative fragment distributions for 3D case are compared in Figure 3.

3. COMPARISON WITH EXPERIMENTS AND DISCUSSION

Applicability of analyzed random fragmentation models will be illustrated through comparison with available experimental results. Having in mind military applications of fragmentation process, all considered experiments are related to the fragmentation of rings and shells of ductile metals.

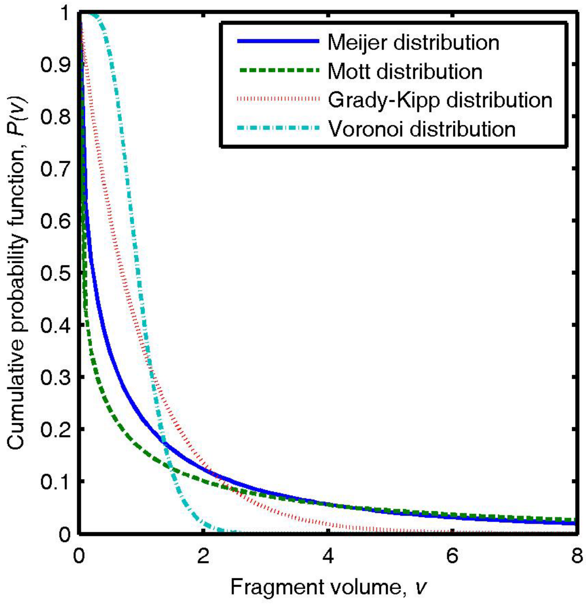

Figure 3. Three-dimensional fragmentation: comparison of the Meijer, Mott, Grady-Kipp and Voronoi model of fragment volume distribution; the average fragment volume is the same for each distribution

For 1D case, the Lineau (exponential), Zhang, modified Zhang and Voronoi distributions are fitted to the experimental data. The comparison of theoretical distributions with experimental fragmentation results presented in [12] and [11] are shown in Figures 4 and 5, respectively.

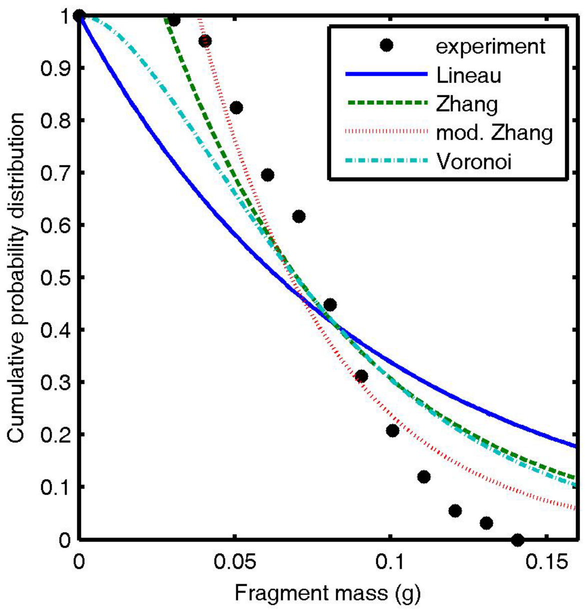

Figure 4. 1D fragmentation: comparison of theoretical distributions with experimental data of aluminum ring fragmentation [12]

Both experiments treat fragmentation of aluminum rings by impulsive electromagnetic loading. Experimental fragmentation data of copper shells induced by gas gun technique [4] are compared to the model results in Figure 6. Finally, the results of 1D fragmentation of magnetically driven uranium-6%-niobium (U6N) rings [21] and corresponding theoretical distributions are shown in Figure 7. The Lineau distribution fails to describe experimental data, while the Voronoi and Zhang distribution provides better agreement with experiments. However, the modified Zhang distribution systematically yields the best compatibility with experimental data. This result emphasizes the importance of the concept of minimum fragment size in 1D fragmentation in conjunction with applied approach based on random selection of fracture points.

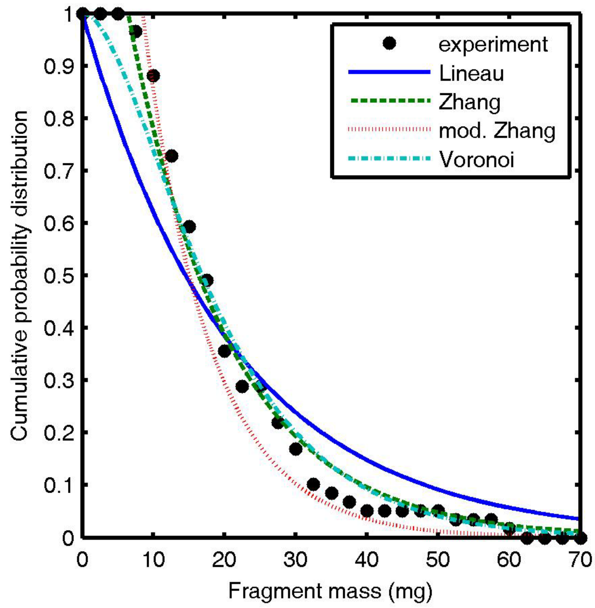

Figure 5. 1D fragmentation: comparison of experimental results of aluminum ring fragmentation [11] with considered theoretical distributions

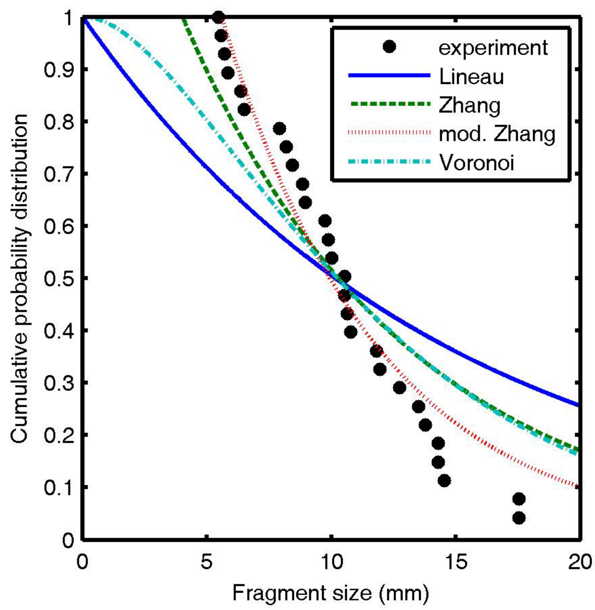

Figure 6. 1D fragmentation: comparison of the theoretical distributions with experimental data of the copper shell fragmentation [4]

Numerically generated fragmentation of a unit square by randomly chosen vertical and horizontal lines (20 x 20) (inset in Figure 8) is compared with analyzed 2D theoretical distributions (with the same average fragment area). This is one possible simulation of 2D fragmentation. As expected, only the Bessel distribution excellently describes numerical data (Fig. 8). Comparison of experimental fragmentation data of the near-spherical ductile metal shell [17] with analyzed theoretical models is presented in Figure 9. Results of explosively induced 2D fragmentation of the stainless steel spherical cap [22] are compared with considered model results (Fig. 10). Both experimental data are fairly approximated by the Voronoi distribution. Possible explanation is that the Voronoi (gamma) fragment size distribution model is approximation of the advanced physically based 2D Mott's fragmentation model [17].

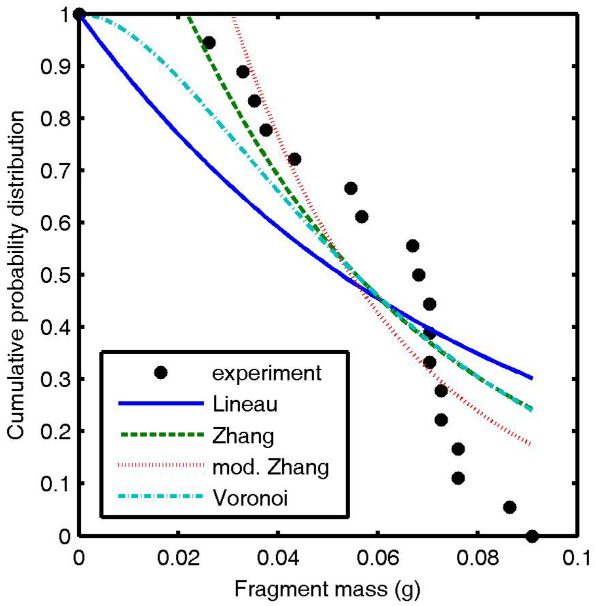

Figure 7. 1D fragmentation: comparison of the experimental results of uranium alloy (U6N) ring fragmentation [21] with considered theoretical distributions

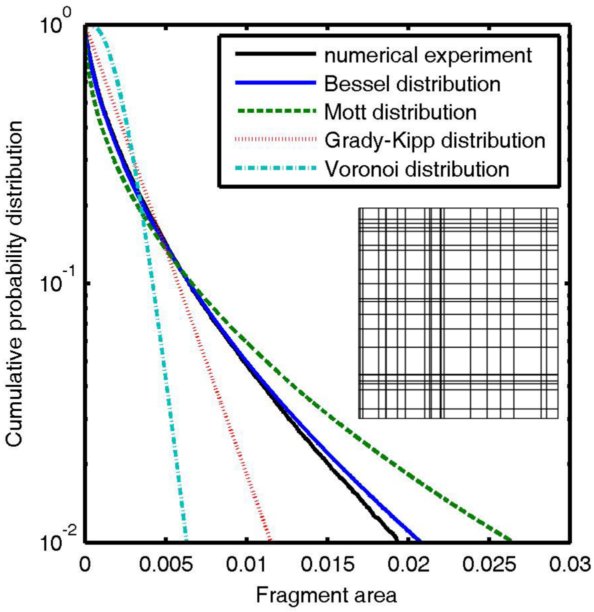

Figure 8. 2D fragmentation: comparison of different theoretical models with numerically determined area distribution obtained by random segmentation of a unit square (inset)

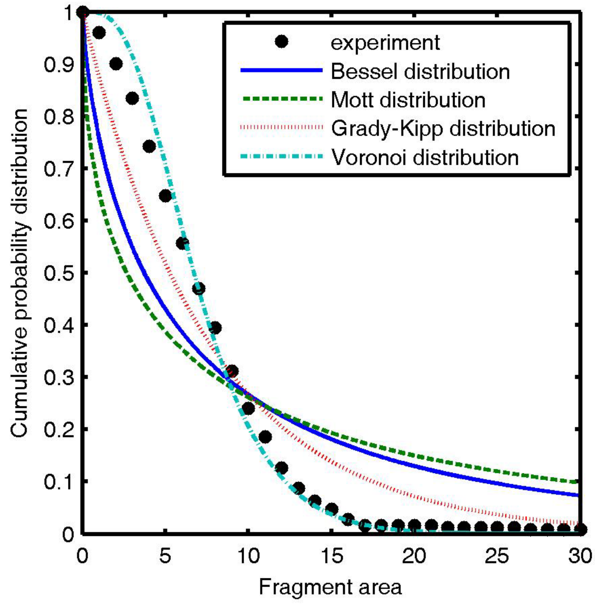

Figure 9. 2D fragmentation: comparison of 2D theoretical models with experimental data from [17]

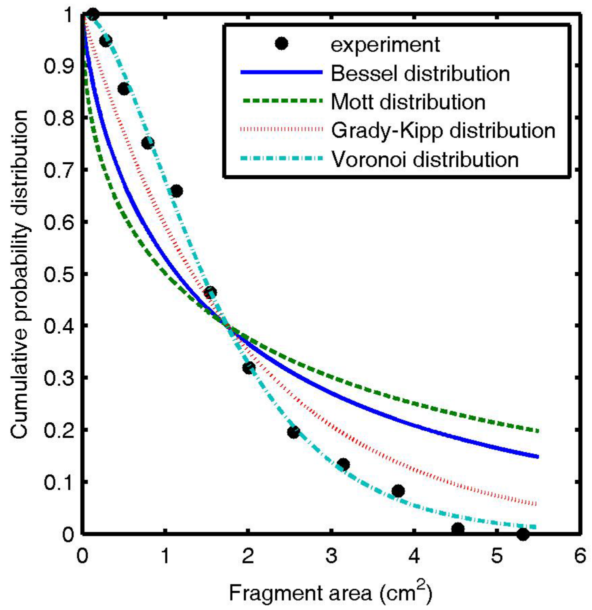

Figure 10. 2D fragmentation: comparison of 2D theoretical models with experimental fragmentation of the spherical cap of steel [22]

Finally, 3D fragmentation experiments are represented by examples of the fragmentation projectile [23] (Fig. 11) and explosively driven steel cylinder [24] (Fig. 12). In the first case, 3D Mott distribution obtains the best fit to experimental data. In the latter, the Meijer and 2D Mott distribution laws have reasonable accordance with experiment. The complexity of 3D fragmentation, which includes tension and adiabatic shearing mechanisms, leads to a variety of fragment sizes and shapes and prevents the analyzed one-parametric distributions to successfully describe the real fragment volume distribution. Advanced treatment of fragment mass distribution in 3D fragmentation using the three-parametric bimodal exponential distribution (generalized Grady-Kipp distribution) is presented in [25].

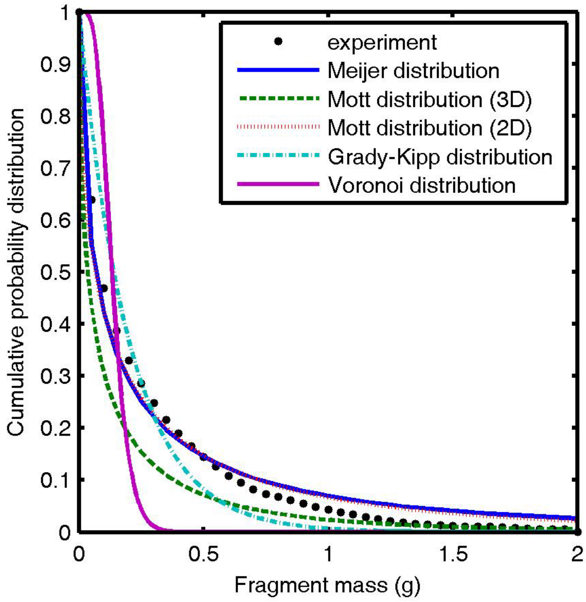

Figure 11. 3D fragmentation: the theoretical distribution laws compared with fragment distribution data of the fragmentation projectile A from [23]

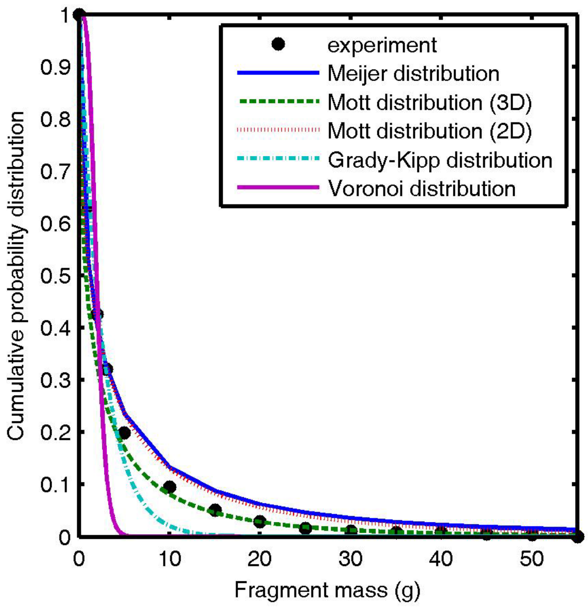

Figure 12. 3D fragmentation: the theoretical distribution laws compared with fragment distribution data of the explosively driven cylinder [24]

4. CONCLUSIONS

The problem of fragment size distribution in dynamic fragmentation is considered from the aspect of geometric probability. We have rederived the Lineau (exponential) fragment distribution law starting from a random segmentation of a line. This approach is generalized to the 2D and 3D case through the Bessel and Meijer distributions. The Mott and Grady-Kipp paradigms are explained and corresponding fragment size distributions are also presented. Also, geometric fragmentation model based on the Voronoi diagrams is introduced.

The analyzed models are compared with limited experimental results related to dynamic fragmentation of ductile metals. In the case of 1D fragmentation, it is shown that the modified Zhang distribution provides the best compatibility with experimental data. Experimental results in 2D fragmentation can be successfully described by the Voronoi distribution. The Mott distributions are the best choice for approximate description of fragment volume distribution in 3D fragmentation.

Having in mind that material characteristics, problem geometry and applied loads are not considered, and that all these parameters are lumped into one adjustable parameter, the results are surprisingly good.

ACKNOWLEDGMENT

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РАСПОДЕЛА ВЕЛИЧИНЕ ФРАГМЕНАТА ПРИ ДИНАМИЧКОЈ ФРАГМЕНТАЦИЈИ: ПРИСТУП ЗАСНОВАН НА ГЕОМЕТРИЈСКОЈ ВЕРОВАТНОЋИ

Предраг Елек, Слободан Јарамаз

Динамичка фрагментација је комплексна појава која карактерише бројне природне и техничке системе. Одређивање закона расподеле величине (односно масе) генерисаних фрагмената представља један од најзначајнијих задатака при моделирању динамичке фрагментације. У раду се разматра уопштен приступ овом проблему заснован на једноставној претпоставци о случајној геометријској сегментацији тела. Полазећи од биномне расподеле места лома (тачака, правих или равни), изведене су функције расподеле величине фрагмената за 1Д, 2Д и 3Д геометрију. Анализирани су модели геометријске фрагментације засновани на приступима *Mott*-а и *Grady-Kipp*-а. Такође су разматрани закони расподеле засновани на примени *Voronoi* дијаграма. Резултати разматраних модела упоређени су са нумеричким симулацијама и експерименталним резултатима, при чему је показано да постоје значајна подударања, као и извесна ограничења модела. Закључено је да преферирани теоријски модел зависи од димензионалности фрагментационог процеса.