Torsional Analysis of Open Section Thin-Walled Beams

The main purpose of this paper is to present one approach to the optimization of thin-walled I, Z and channel-section beams subjected to constrained torsion. The displacement constraints are introduced: allowable angle of twist and allowable angle of twist per unit length. The area of the cross-section is assumed to be the objective function. Applying the Lagrange multiplier method, the equations whose solutions represent the optimal values of the ratios of the parts of the chosen cross-sections are derived.

**Keywords:** thin-walled beams, cantilever beam, optimal dimensions, displacement constraints.

1. INTRODUCTION

Many studies have been made on the optimization problems treating the cases where geometric configurations of structures are specified and only the dimensions of members, such as areas of members’ cross-sections, are determined in order to attain the minimum structural weight or cost [1-5]. Many methods have been developed for the determination of the local minimum point for the optimization problem [6-8]. Very often used types of cross-sections, particularly in steel structures are the I, Z and channel sections.

A series of works appear where the optimization parameters of various cross-sections, such as I-section [9], channel-section [10] or Z-section beams [11] have been determined by Lagrange’s multipliers method.

The starting points during the formulation of the basic mathematical model are the assumptions of the thin-walled beam theory, on one hand [12, 13], and the basic assumptions of the optimum design on the other [1-5].

Open thin-walled steel sections subjected to twisting moments are generally prone to large warping stresses and excessive angles of twist. It is therefore a common practice to avoid twisting moments in steel assemblies consisting of steel open sections whenever it is possible. However, in a number of practical applications, twisting cannot be avoided and the designer is compelled to count on the torsional resistance of these members. The classical formulation for open thin-walled sections subjected to torsion was developed by Vlasov [13]. The Vlasov formulation is based on two fundamental kinematic assumptions: (a) In-plane deformations of the section are negligible, and (b) shear strains along the section mid-surface are negligible.

2. BASIC ASSUMPTIONS

The formulation is restricted to the torsional analysis of open section thin-walled beams.

The considered cantilever beam, of the length \( l \) is subjected to the constrained torsion because of the fact that its one end is fixed and the other free end is loaded by a concentrated torque \( M^* \). The cross-section (Fig. 1) is supposed to have flanges of mutually equal widths and thicknesses \( b_1 = b_3, t_1 = t_3 \).

2.1 Objective function

It is very important to find out the optimal dimensions of cross-sections. The process of selecting the best solution from various possible solutions must be based on a prescribed criterion known as the objective function. In the considered problem the cross-sectional area will be treated as an objective function. The aim of the paper is to determine the minimal mass of the beam or, in another way, to find the minimal cross-sectional area

\[
A = A_{\text{min}}
\]

for the given loads and material and geometrical properties of the considered beam.

It is obvious from the Fig. 1 that

\[
A = \sum b_it_i, \quad i = 1,2,3
\]

where \( b_i \) and \( t_i \) are widths and thicknesses of the parts of the considered cross-sections.

2.2 Constraints

Only the displacements will be taken into account in the calculations that follow and the constraints treated in the paper are the displacement constraints.
The considered displacement constraints are allowable angle of twist and allowable angle of twist per unit length denoted by \( \theta_0 \) and \( \theta_0' \) respectively.

The ratio

\[
z = \frac{b_2}{h_t}
\]

will be the optimal relation of the dimensions of the considered cross-sections.

The flexural-torsion cross-section characteristic [5] is given by the expression

\[
k = \sqrt{G I_t / E I_{\omega}}
\]

where:
- \( \psi = t_2 / t_1 \) - the ratios of thickness and length of the parts of the cross-section
- \( I_t \) - torsion constant derived for each of the three considered cross-sections [12],
- \( I_{\omega} \) - sectorial moment of inertia derived for each of the three considered cross-sections [12],
- \( E \) - modulus of elasticity and
- \( G \) - shear modulus.

If the constraint is allowable angle of twist \( \theta_0 \), the constraind function (5)

\[
\theta_{\text{max}} = \theta(l) = \frac{M^* I}{G I_t} \left(1 - \frac{1}{C h k l} \right) \leq \theta_0
\]

can be written as

\[
\varphi_1 = kl - Th kl - \frac{G I_t}{M^*} \theta_0 \leq 0.
\]

If the constraint is allowable angle of twist per unit length \( \theta_0' \), the constraind function (7)

\[
\theta'_{\text{max}} = \dot{\theta}(l) = \frac{M^*}{G I_t} \left(1 - \frac{1}{C h k l} \right) \leq \theta_0'
\]

can be written as

\[
\varphi_2 = Ch kl \left(1 - \theta_0' \frac{G I_t}{M^*} \right) - 1 \leq 0.
\]

### 2.3 Lagrange multiplier method

The Lagrange multiplier method [2,3,14,15] is a powerful tool for solving this class of problems and represents the classical approach to the constraint optimization.

Applying the Lagrange multiplier method to the vector which depends on two parameters \( b_i \), \( i = 1, 2 \), the system of equations \( \varphi(b_i) = 0 \), \( i = 1, 2 \) will be obtained

\[
\frac{\partial}{\partial b_i} \left[ A(b_1, b_2) + \lambda \varphi(b_1, b_2) \right] = 0, \quad (i = 1, 2)
\]

and after the elimination of the multiplier \( \lambda \) from (9), it becomes (10)

\[
\frac{\partial A(b_1, b_2)}{\partial b_1} + \frac{\partial \varphi(b_1, b_2)}{\partial b_1} = 0
\]

\[
\frac{\partial A(b_1, b_2)}{\partial b_2} + \frac{\partial \varphi(b_1, b_2)}{\partial b_2} = 0
\]

### 3. ANALYTICAL APPROACH

The diagrams of sectorial coordinates [5] are shown in the Fig.2 for each cross-section separately, where P is the shear center.

#### 3.1 I cross-section

The expressions of torsion constant and sectorial moment of inertia for the I-section are:

\[
I_t = \frac{1}{3} b_1 b_2^3 \left(2 + \psi^3 z^2 \right), \quad I_{\omega} = \int_A \omega^2 dA
\]

Applying the Lagrange multiplier method, the equations (13) according to allowable angle of twist \( \theta_0 \), and (14) according to allowable angle of twist per unit length \( \theta_0' \) are obtained:

- \( \theta_0 \) – constraint:

\[
2 \frac{G I_t}{E k} \frac{1}{b_1^2 b_2^3} \left[ -3 T h k l + G I_t b_1 t_1 \left(2 + \psi^3 z \right) \right]
\]

\[
\cdot \left(8 - 4 \psi z + 2 \psi^3 z - 3 \psi^4 z^2 \right) - \frac{G I_t}{M^*} \left(84 - 4 \psi z + 2 \psi^3 z - 3 \psi^4 z^2 \right) + \frac{G I_t}{M^*} \left(1 - \psi^2 \right) \psi z = 0
\]

- \( \theta_0' \) – constraint:

\[
2 T h k l \frac{1}{b_1^2 b_2^3} \frac{G I_t}{E k} \left[ 3 - \frac{G I_t}{M^*} b_1 t_1 \left(2 + \psi^3 z \right) \right]
\]

\[
- \left(84 + 4 \psi z + 2 \psi^3 z + 3 \psi^4 z^2 \right) + \frac{G I_t}{M^*} \left(1 - \psi^2 \right) \psi z = 0.
\]
In the considered case when the $I$ – beam is the object of the optimization the equations (13) and (14), combined with (6) and (8) are reduced to the equation

$$\sum_{i=0}^{2} c_i z^i = 0$$  \hspace{1cm} (15)

where the coefficients $c_i$ are given in the form (16) if the constraint is allowable angle of twist ($\theta_0$), i.e. (17) if the constraint is allowable angle of twist per unit length ($\theta_0'$):

- $\theta_0$ – constraint:

$$c_0 = 8,$$

$$c_1 = -2\psi \left[ 2 - \psi^2 + 2 \frac{\psi^2 - 1}{kl - kT \varphi^2 - kT \varphi l} \right],$$

$$c_2 = -3\psi^4.$$  \hspace{1cm} (16)

- $\theta_0'$ – constraint:

$$c_0 = -8,$$

$$c_1 = 2\psi \left[ 2 - \psi^2 + 2 \frac{\psi^2 - 1}{kl - kT \varphi^2 - kT \varphi l} \right],$$

$$c_2 = 3\psi^4.$$  \hspace{1cm} (17)

### 3.2 Z cross-section

The expressions of torsion constant and sectorial moment of inertia for the Z-section are:

$$I_t = \frac{1}{3} b_1^3 t_3 \left( 2 + \psi^3 z \right), I_{c0} = \frac{1}{12} b_1^3 b_2^2 \frac{1 + 2\psi z}{2 + \psi z}$$  \hspace{1cm} (18)

Applying the Lagrange multiplier method, the equations (19) and (20) are obtained

- $\theta_0$ – constraint:

$$4 \frac{Gl}{E k} \frac{t_1^2}{b_1^2 b_2^2} \left( 8 + 22\psi z + 2\psi^3 z + \psi^2 z^2 + 5\psi^4 z^2 \right) +$$

$$\left( 1 + 2\psi z \right)^2 +$$

$$\frac{-4\psi^3 z^3 - 4\psi^5 z^3 - 3\psi^6 z^4}{\left( 1 + 2\psi z \right)^2} =$$

$$-Th^2 kl + \frac{1}{3} \frac{Gl t_1}{M^*} b_1 t_1^3 \left( 2 + \psi^3 z \right) -$$

$$\frac{-2}{3} \frac{G l}{M^*} kl t_1^3 \left( \psi^2 - 1 \right) \psi z = 0.$$  \hspace{1cm} (19)

- $\theta_0'$ – constraint:

$$\sum_{i=0}^{4} c_i z^i = 0,$$  \hspace{1cm} (21)

where the coefficients $c_i$ are:

- $\theta_0$ – constraint:

$$c_0 = 8,$$

$$c_1 = 2\psi \left[ 11 + \psi^2 - 2 \frac{\psi^2 - 1}{kl t_1^2 kl - kT \varphi^2 - kT \varphi l} \right],$$

$$c_2 = -4\psi^3 \left[ 1 + 5\psi^2 - \frac{\psi^2 - 1}{kl T \varphi^2 kl - kT \varphi l} \right],$$

$$c_3 = -3\psi^6.$$  \hspace{1cm} (23)

- $\theta_0'$ – constraint:

$$c_0 = -8,$$

$$c_1 = -2\psi \left[ 11 + \psi^2 - 2 \frac{\psi^2 - 1}{kl T \varphi^2 kl - kT \varphi l} \right],$$

$$c_2 = -4\psi^3 \left[ 1 + 5\psi^2 - \frac{\psi^2 - 1}{kl T \varphi^2 kl - kT \varphi l} \right],$$

$$c_3 = 3\psi^6.$$  \hspace{1cm} (23)
3.3 Channel cross-section

The expressions of torsion constant and sectorial moment of inertia for the channel-section are:

\[ I_t = \frac{1}{3} b_1 t_1^3 \left( 2 + \psi^3 z \right), \]

\[ I_{\omega} = \frac{1}{12} b_1^3 t_1^2 \frac{1 + 2\psi z}{2 + \psi z}. \]  

(24)

Applying the Lagrange multiplier method, the equations (25) and (26) are obtained.

\[ -\theta_0 \text{ constraint:} \]

\[ 4 Gl \frac{l_t^2}{E k b_1^2 b_2^2} \]

\[ \sum \left[ 72 + 42\psi z + 18\psi^3 z - 13\psi^2 z^2 - 3\psi^4 z^2 + \frac{4\psi^3 z^3 - 16\psi^5 z^3 - 3\psi^6 z^4}{(3 + 2\psi z)^2} + \frac{-4\psi^3 z^3 - 16\psi^5 z^3 - 3\psi^6 z^4}{(3 + 2\psi z)^2} \right] \]

\[ \cdot \left[ 2Thkl \frac{G\theta_0}{3M^*l} b_1 t_1^3 \left( 2 + \psi^3 z \right) \right] - \]

\[ \frac{2}{3 M^*l} \psi z b_1 t_1^3 \left( \psi^2 - 1 \right) = 0. \]  

(25)

\[ \theta_0' \text{ constraint:} \]

\[ 2Thkl \frac{Gl t_1^2}{E k b_1^2 b_2^2} \]

\[ \sum \left[ -72 - 42\psi z - 18\psi^3 z + 13\psi^2 z^2 + 3\psi^4 z^2 + \frac{4\psi^3 z^3 + 16\psi^5 z^3 + 3\psi^6 z^4}{(3 + 2\psi z)^2} \right] \]

\[ \cdot \left[ 1 - \frac{1}{3 M^*l} b_1 t_1^3 \left( 2 + \psi^3 z \right) \right] + \]

\[ + \frac{2}{3 M^*l} \psi z b_1 t_1^3 \left( 1 - \psi^2 \right) = 0. \]  

(26)

In the considered case when the channel-section is the object of the optimization the equations (25) and (26), combined with (6) and (8) are reduced to the equation

\[ \sum_{i=0}^{4} c_i z^i = 0 \]  

(27)

- \theta_0 \text{ constraint:} \]

\[ \sum_{i=0}^{4} c_i z^i = 0 \]

\[ \begin{align*}
    c_0 &= 72, \\
    c_1 &= 6\psi \left[ 7 + 3\psi^2 - 6 \frac{\psi^2 - 1}{kl Thhl^2 kl} \right], \\
    c_2 &= -\psi^2 \left[ 13 + 3\psi^2 + 30 \frac{\psi^2 - 1}{kl Thkl} \right], \\
    c_3 &= -4\psi^3 \left[ 1 + 4\psi^2 + \frac{\psi^2 - 1}{kl Thkl} \right], \\
    c_4 &= -3\psi^6. 
\end{align*} \]  

(28)

- \theta_0' \text{ constraint:} \]

\[ \sum_{i=0}^{4} c_i z^i = 0 \]

\[ \begin{align*}
    c_0 &= -72, \\
    c_1 &= -6\psi \left[ 7 + 3\psi^2 - 6 \frac{\psi^2 - 1}{1 - Chkl} \right], \\
    c_2 &= \psi^2 \left[ 13 + 3\psi^2 + 30 \frac{\psi^2 - 1}{1 - Chkl} \right], \\
    c_3 &= 4\psi^3 \left[ 1 + 4\psi^2 + \frac{\psi^2 - 1}{1 - Chkl} \right], \\
    c_4 &= 3\psi^6. 
\end{align*} \]  

(29)

4. ANALYSIS AND DISCUSSIONS

The following expressions will be introduced

\[ D = \frac{\psi^2 - 1}{kl Thhl - kl Thkl}, \]

\[ D_t = \frac{\psi^2 - 1}{1 - Chkl}. \]  

(30)

The calculation is made for the cantilever beam of chosen section of the length 0.25 ≤ l ≤ 200 cm. Values \( kl \) are calculated using data for standard profiles and ratio \( \psi = t_2/t_1 \) is taken as \( \psi = 0.5; 0.75; 1 \).

The results for ratios (3) \( z = b_2/b_1 \) are presented graphically in Figs. 3-5.
After the calculations, it can be concluded that the increase of the expressions $D_1$, e.i. $D_1$, will decrease the optimal relations $z$.

**I – beam:**
- Optimal values $z$ for strain constraint $\theta_0$
  - $\psi=1 \Rightarrow D_1 = 0 \Rightarrow z = \text{const} = 1.33$,
  - $\psi=0.75 \Rightarrow 0.22 \leq D_1 \leq 437.5 \Rightarrow z \geq 1.78 \geq 0$,
  - $\psi=0.5 \Rightarrow 0.38 \leq D_1 \leq 750 \Rightarrow 2.67 \geq z \geq 0$.

The calculations show that the optimal values of $z$ for the I-section beam are very small for the lengths $l > 100 \text{ cm}$. Because of that it is possible to say that the application of this criterion makes sense for following lengths:

- for $\psi=0.5$: $l \approx 90 \text{ cm} \Rightarrow z \geq 0.51$, and
- for $\psi=0.75$: $l \approx 95 \text{ cm} \Rightarrow z \geq 0.45$.

**Z – beam:**
- Optimal values $z$ for strain constraint $\theta_0$

  The optimal values of $z$ are for the lengths $l \leq 100 \text{ cm}$
  - $\psi=1 \Rightarrow D_1 = 0 \Rightarrow z = \text{const} = 1.72$,
  - $\psi=0.75 \Rightarrow l \approx 150 \text{ cm}, D_1 \leq 5 \Rightarrow z \geq 0.4532$,
  - $\psi=0.5 \Rightarrow l \approx 120 \text{ cm}, D_1 \leq 8 \Rightarrow z \geq 0.50688$.
The optimal values of \( z \) are for the lengths \( l \leq 150 \text{ cm} \).

- Optimal values \( z \) for strain constraint \( \theta_0 \)
  - \( \psi = 1 \Rightarrow D_1 = 0 \Rightarrow z = \text{const} = 1.72 \),
  - \( \psi = 0.75 \): \( l = 48 \text{ cm}, D_1 \leq 5 \Rightarrow z \geq 0.5372 \),
  - \( \psi = 0.5 \): \( l = 47 \text{ cm}, D_1 \leq 8 \Rightarrow z \geq 0.50688 \).

The optimal values of \( z \) are for the lengths \( l \leq 50 \text{ cm} \). Channel section – beam:

Calculations show that optimal values \( z \), for the constraints \( \theta_0 \) and \( \theta_0 \), are for the lengths \( l \leq 200 \text{ cm} \).

5. CONCLUSION

This paper presents an approach to the optimization of thin-walled open-section cantilever beams, using the Lagrange multiplier method. Selecting the cross-section area as the objective function and deformation constrains for constraint functions, optimal ratios of cross-section individual parts (webs and flanges) are determined.

Based on the obtained results (Figs. 3, 4 and 5), it can be seen that some differences exist between coefficients \( c_i \), calculated using the criteria \( \theta_0 \) or \( \theta_0 \), and a minimum disagreement between obtained values for \( z \) is observed. Optimal values \( z \) obtained by using criterion the \( \theta_0 \) are slightly higher than values obtained by the \( \theta_0 \) criterion.

On the bases of the proposed optimization procedure, it is possible to calculate the optimal ratios between the parts of the considered thin-walled profiles in the a very simple way.

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REFERENCES


ТАНКОЗИДИ ОТВОРЕНИ ПОПРЕЧНИ ПРЕСЕЦИ ИЗЛОЖЕНИ ОГРАНИЧЕНОЈ ТОРЗИЈИ

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Основни циљ овог рада је да прикаже један приступ оптимизацији танкозидних I, Z и U козолих конструкција изложеном површином поперчног пресека и ограниченој тора зији. За критеријум ограничења одабран је критеријум ограничења деформација: дозвољени угао увијања и дозвољене уга увијања по јединици дужине. За функцију циља одабрана је површина поперчног пресека носача. Применом методе Лагранжовог множиета изведене су јединице чија решења представљају оптималне односно димензије поперчног пресека изабраног облика.