Kinematics of Base Sphere as a Segment of Spheres Kinematical Chain - Gravitational acceleration as equivalent central acceleration

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Assuming that the theory for differential geometry of curve is correct, this paper introduces a spheres kinematical chain of changeable radii positioned in such a fashion that the centre of the next sphere is firmly attached to the previous one within the chain. The kinematical chain consists of the base sphere and the rest of the chain which is inside the base sphere. The paper discusses the undisturbed and disturbed motion of the base sphere. Tha gravitational acceleration is interpreted as equivalent central acceleration during undisturbed rotation of the base sphere. The disturbed motion with respect to the undisturbed motion is described by increments of corresponding kinematical quantities. The relationship with the existing laws for translational motion in the classical mechanics is also established. And finally, a simple experiment for verification of the exposed theory is proposed in this paper.

Keywords: *kinematical chain, sphere of changeable radius, spheres rotation, gravitational acceleration as equivalent central acceleration, increments of velocity and acceleration of relative disturbed motion, translational motion.*

1. INTRODUCTION

Current science and technology level give us the means to solve the most complicated technical tasks. This is also the case for many explanations of gravity and motion in space (enough to see the theory review on web page http://en.wikipedia.org/wiki/Gravitation).

The problem arose when the researchers tried to explain the functioning of living beings. One of the main tasks of contemporary robotics is to determine the mechanical principles upon which the dynamics, walking control and flight of living beings, especially two-legged ones, are based. Knowing the functioning principles of living beings and their balance centres would enable a proper choice of possible technical solutions for particular purpose robots. Some consistent approaches for solving these problems have been adopted, out of which the most frequently used are the ZMP method [1] and method based on the relation $\mathbf{a} = \mathbf{t} - \mathbf{g}$, $d\mathbf{g}/d\mathbf{t} = \boldsymbol{\omega} \times \mathbf{g}$, where $\mathbf{a}$ is the net acceleration sensed by the otoliths, $\mathbf{t}$ translational and $\mathbf{g}$ gravitational acceleration, and $\boldsymbol{\omega}$ angular velocity, e.g. [2]. All living beings possess motion detection sensors . Information obtained by these sensors are used for keeping balance and controlling the motions in space of all types. These information are being gathered during disturbed motion within the coordinate frame firmly attached to the body. Motion sensors of living beings detect the increment of kinematical quantities, not their absolute value. Positive acceleration along the longitudinal axis of a living beings body is estimated as acceleration which is directed from foot to head, that is, upward [3],[4], which is in contrast with the adopted direction of the gravitational acceleration.

The exact analysis of known mathematical results for the differential geometry of curve gives us new interpretation for motion of a point along a curve as function of rotation and deformation of spheres which are part of a kinematical chain of spheres [5]. In this paper the kinematical chain of spheres is composed of the base sphere and the rest of the kinematical chain of spheres. The rest of the kinematical chain is placed in the interior of the base sphere. The paper discusses the motion of the base sphere and in that sense three tasks present themselves. The first one is to present the gravitational acceleration as equivalent central acceleration as a consequence of undisturbed base sphere rotation. The undisturbed rotation of the base sphere implies that it is rotating at angular velocity of constant direction with respect to the immobile environment or environment rotating at a slower rate provided that its centre is still within the environment. As a part of this task we found a relationship that can lead to important conclusions about the direction and character of the gravitational acceleration. The second task is to obtain accurate kinematical relations for velocity and acceleration of points during disturbed motion of the base sphere. The disturbed motion of the base sphere implies its rotation such that there are increments in the vector of angular velocity and sphere radius compared to the undisturbed motion. The aim is to obtain accurate and approximate relations for absolute velocity and acceleration of a point in the

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kinematical chain of spheres. The third task is to establish the relationship with the existing relations for translational motion found in classical mechanics, based on the results obtained by completing the previous two tasks. Finally, the motion of objects under the influence of gravitational acceleration as equivalent central acceleration will be considered. It will be shown that there is no contradiction with classical theory of mechanics. Also, there is a suggestion for a simple experiment that can verify existing theories about the Earth's gravity and the theory of gravity as the central acceleration.

2. GEOMETRICAL AND KINEMATICAL THEORETICAL BASIS

Consider an arbitrary spatial curve P_e and choose arbitrary point E on it.

In mathematics it is known [6] that a curve element around the point E lies wholly on the sphere of radius $\overline{R}=\sqrt{\rho_k^2+(\rho_\tau \rho'_k)^2}$, where and $\rho_\tau = 1/\tau$ are the radius of curvature and radius of torsion respectively, ρ'_k is the derivative of the radius of curvature with respect to the arc length, τ and k are torsion and curvature of the curve P_e respectively. The center of the sphere is located at the line which is parallel to the binormal and passes through the center of curvature of the curve P_e.

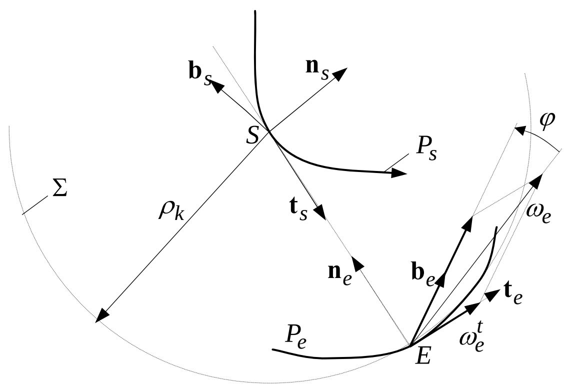

Figure 1. Motion of point and curve parameters

Consider a sphere whose center is on the principal normal of the curve and whose radius is equal to the radius of curvature at E, which is $\rho_k = 1/|k|$. The natural trihedron of the curve with unit vectors $\mathbf{t}_e$, $\mathbf{n}_e$, $\mathbf{b}_e$ lies on this sphere with origin at the point E. The ort $\mathbf{n}_e$ is oriented towards the center of the sphere, and ort $\mathbf{t}_e$ is tangent to the curve. During the motion of the point E along the curve, the radius of curvature and orientation of the natural trihedron are changing. The cruising angular velocity (Darboux vector) of the natural trihedron is the vector $\boldsymbol{\omega}_\mathbf{e}^\mathrm{s} = \begin{pmatrix}\tau & 0 & k\end{pmatrix}\underline{\mathbf{e}}_{en}$ [6], [7], where $\underline{\mathbf{e}}_{en} = \begin{pmatrix}\mathbf{t}_e & \mathbf{n}_e & \mathbf{b}_e\end{pmatrix}^\mathrm{T}$. If the motion of the point E is observed with respect to time, then angular velocity $\boldsymbol{\omega}_\mathbf{e}$ of the natural trihedron is obtained as the product of its cruising angular velocity $\boldsymbol{\omega}_\mathbf{e}^\mathrm{s}$, and the speed $V = |\mathbf{V}_\mathbf{e}|$ of the velocity $\mathbf{V}_\mathbf{e} = (V \quad 0 \quad 0)\underline{\mathbf{e}}_{en}$ of the point E, $\boldsymbol{\omega}_\mathbf{e} = V\cdot\boldsymbol{\omega}_\mathbf{e}^\mathrm{s} = (\omega_e^t \quad 0 \quad \omega_e^b)\underline{\mathbf{e}}_{en} = (\tau V \quad 0 \quad kV)\underline{\mathbf{e}}_{en}$ [7], [8],. During this motion, the acceleration of the point E is $\mathbf{a}_\mathbf{e} = d_{en}\mathbf{V}_\mathbf{e}/dt + \boldsymbol{\omega}_\mathbf{e}\times\mathbf{V}_\mathbf{e} = (a_e^t \quad a_e^n \quad 0)\underline{\mathbf{e}}_{en} = (\dot{V} \quad kV^2 \quad 0)\underline{\mathbf{e}}_{en}$ where $d_{en}(\cdot)/dt$ is the relative derivative of a vector in the natural trihedron $\underline{\mathbf{e}}_{en}$. If the curve is defined, using Frenet–Serret formulas it is easy to find the curvature k and torsion τ of the curve and orts of the natural trihedron. If the magnitude V is known, then the motion of the point E is defined up to the velocity of the point E. It means that if the parameters of curve, the curvature k and torsion τ, are known, then there are infinitely many curves in space with these parameters. Therefore, the mathematical analysis of motion of the point E is based on the defined curve P_e (by parameters k and τ, and a position of E on the curve), and defined speed V along the curve.

We shall remove the path P_e and consider the motion of the sphere Σ that contains the point E, while its center is at the center of curvature S of the path P_e at point E. .It is assumed that the sphere Σ is firmly attached to the natural trihedron at point E. The motion of the point E and motion of the natural trihedron of the path can be considered as a consequence of pure deformation and pure rotation of the sphere Σ to which this point is firmly attached.

The center S of the sphere Σ travels along a curve P_s. Analogously, the motion of a sphere that presents the motion of the center S along the path P_s can be defined. By this way, a kinematical chain of spheres of variable radii is formed. Absolute motion of any point E, as a point at the last sphere in the chain, can be produced by the movement of the chain.

3. KINEMATICAL CHAIN OF SPHERES

Having in mind the previously said and the fact that the skeleton of almost all living beings is a mechanism with spherical joints, we introduce the kinematical chain of spheres as kinematical model of a mechanical system moving on the Earth’s surface. The chain is formed such that every radius vector is considered the position vector of a point on the sphere whose radius is equal to the magnitude of the position vector and whose center is at the origin of the position vector (Fig. 2).

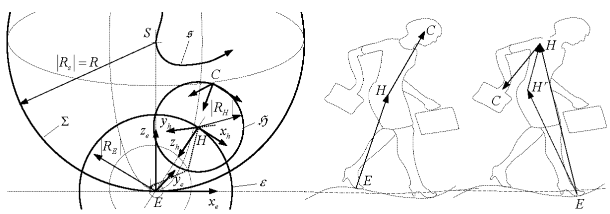

Figure 2: Spheres kinematical chain

The spheres are positioned in such a fashion that the centre of the next sphere is firmly attached to the previous one within the chain. The centre S of the base sphere Σ whose radius is $R_S = R$ is either still or it moves along a trajectory s. The sphere Σ rotates with the angular velocity $\boldsymbol{\omega}_\mathbf{e}$. The centre E of the sphere ε with radius R_E is attached to the sphere Σ. The centre H of the sphere $\mathcal{H}$ of radius R_H is attached to the sphere ε. The centre C of a new sphere is then attached to the sphere $\mathcal{H}$ and so on. All spheres' centres starting with the centre H are within the sphere Σ. The spheres ε and $\mathcal{H}$ rotate at angular velocities $\Delta\ddot{\overline{\mathbf{R}}}_\mathbf{ee} = d_r(\Delta\dot{\mathbf{R}}_\mathbf{ee})/dt$ and $\boldsymbol{\omega}_\mathbf{c}$ successively with respect to the spheres Σ and ε. The radii of the spheres and their angular velocities can be changeable during the motion. Coordinate frames are introduced so that axes x and y determine the tangential plane of a sphere, while the third axis is always pointed towards the centre of the sphere.

An example of such a chain can be a human body. The point E is contact point or ZMP, while other points on spheres in the chain are spherical joints and also, a last observed point of the body. For example, the point H can be a point on hip, the point C may be mass center or a point on/in head. In case of an object in hand, the point H can be a spherical joint of shoulders, while the point C could be a point at hand. In this case, two spheres, whose radii are EH' (contact point E - spherical joint of hip H') and $H'H$ (spherical joint of hip H' - spherical joint of shoulder H), may replace the sphere of radius EH (contact point E - spherical joint of shoulder H).

4. GRAVITATIONAL AS EQUIVALENT CENTRAL ACCELERATION - UNDISTURBED MOTION

If we set an arbitrary coordinate frame at the point E, the projections of its absolute angular velocity and projections of the absolute linear acceleration $\mathbf{a}_\mathbf{e}$ will be measured. If the coordinate frame does not rotate with respect to the natural trihedron at E then its angular velocity will be the absolute angular velocity of the natural trihedron According to the approach presented here, acceleration exists only if the point moves nonuniformly along a curve, that is, if there is a deformation and/or rotation of a sphere in an inertial coordinate system. From this we conclude that gravitational acceleration must be result of motion of a sphere in an inertial coordinate system (Fig. 3). In this paper we are focused to motions of the base sphere Σ.

While analyzing motion of bodies on the Earth's surface, the motion of the Earth is taken to be approximately inertial. If it is presumed that the gravitational acceleration is the result of motion along a curve then the motion of the Earth is non-inertial. Therefore, motions described in a coordinate frame attached to the Earth are motions seen from non-inertial coordinate frame which, in no way, can be considered approximately inertial. Further considerations in the paper assume this fact.

Let a point S be an immobile point. Let a point E be a point on the Earth and let it be immobile with respect to the Earth. Let two spheres Σ_0 and Σ of the same radius R be attached to the centre S (Fig. 3). Let the sphere Σ_0 be still with respect to the immobile environment while the sphere Σ rotates around the centre S. Let $Sx_sy_sz_s$ be a coordinate frame with unit vectors $\underline{\mathbf{e}}_s = (\mathbf{i}_s \quad \mathbf{j}_s \quad \mathbf{k}_s)^\mathrm{T}$ immobile with respect to the fixed sphere Σ_0 and $Ex_ey_ez_e$ a coordinate frame with unit vectors $\underline{\mathbf{e}}_e = (\mathbf{i}_e \quad \mathbf{j}_e \quad \mathbf{k}_e)^\mathrm{T}$ attached to the mobile sphere Σ. Let the axes Sz_s of two coordinate frames match at the initial instant.

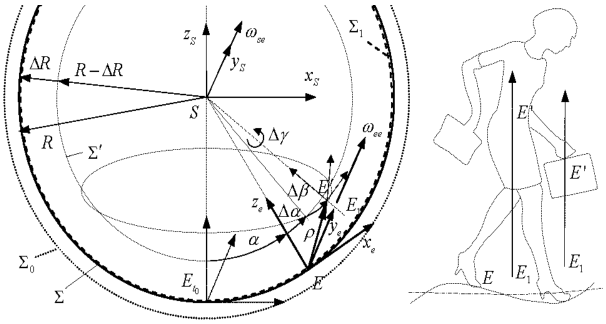

Figure 3: Motion around a fixed centre

Let the undisturbed motion of the sphere Σ be the motion when it rotates around the fixed centre S at angular velocity $\boldsymbol{\omega}_\mathbf{se} = (0 \quad \omega_{se}^y \quad 0)\underline{\mathbf{e}}_s$ co-linear with the axis Sy_s (Fig. 3) and let its radius R change. Let the position, velocity and acceleration of the origin S with respect to the point E be defined by vectors $\mathbf{R}_\mathbf{ee} = \overrightarrow{\mathbf{ES}} = (x_{ee} \quad y_{ee} \quad z_{ee})\underline{\mathbf{e}}_e$, $\mathbf{V}_\mathbf{ee} = \dot{\mathbf{R}}_\mathbf{ee} = (v_{ee}^x \quad v_{ee}^y \quad v_{ee}^z)\underline{\mathbf{e}}_e$, $\mathbf{a}_\mathbf{ee} = \ddot{\mathbf{R}}_\mathbf{ee} = (a_{ee}^x \quad a_{ee}^y \quad a_{ee}^z)\underline{\mathbf{e}}_e$ For undisturbed motion, when the axis x has the same direction as the velocity vector, the following relations apply

$$\begin{array}{lll} x_{ee} = 0 & v_{ee}^x = R\dot{\alpha} & a_{ee}^x = 2\dot{R}\dot{\alpha} + R\ddot{\alpha} \\ y_{ee} = 0 & v_{ee}^y = 0 & a_{ee}^y = 0 \\ z_{ee} = R & v_{ee}^z = \dot{R} & a_{ee}^z = \ddot{R} - R\dot{\alpha}^2 \end{array} \tag{1}$$

where: $\alpha = \int_{t_0} \omega_{se}^y dt$, $\dot{\alpha} = \omega_{se}^y$, $\ddot{\alpha} = \dot{\omega}_{se}^y$, $R = | R_{ee} |$.

The vectors of the angular velocity and acceleration of the coordinate frame $Ex_ey_ez_e$ are

$$\boldsymbol{\omega}_\mathbf{ee} = (0 \quad \dot{\alpha} \quad 0)\underline{\mathbf{e}}_e, \quad \dot{\boldsymbol{\omega}}_\mathbf{ee} = (0 \quad \ddot{\alpha} \quad 0)\underline{\mathbf{e}}_e \tag{2}$$

A special case of the undisturbed motion of the sphere Σ is the motion when the point E has constant speed down a geodesic on the fixed sphere Σ_0 of approximately constant radius. In that case $R = const$ and $\dot{\alpha} = const$, so equations (1) amount to

$$\begin{array}{lll} x_{ee} = 0 & v_{ee}^x = R\dot{\alpha} & a_{ee}^x = 0 \\ y_{ee} = 0 & v_{ee}^y = 0 & a_{ee}^y = 0 \\ z_{ee} = R & v_{ee}^z = 0 & a_{ee}^z = -R\dot{\alpha}^2 \end{array} \tag{3}$$

It is known from mathematical references [9] that a geodesic is the shortest curve connecting two points on a surface. Its characteristic is that its normal is parallel to the main normal of the surface. The coordinate frame $Ex_ey_ez_e$ is therefore the natural trihedron of the geodesic at point E, and the axis Ez_e is the axis of the normal.

From equation (3) follows that the acceleration $\ddot{\mathbf{R}}_{\mathbf{ee}}$ lies wholly along the axis Ez_e and is oppositely directed to it. If the term $R\dot{\alpha}^2$ equals the magnitude of the gravitational acceleration the acceleration $\ddot{\mathbf{R}}_{\mathbf{ee}}$ is equal to the currently adopted gravitational acceleration. However, the acceleration $\ddot{\mathbf{R}}_{\mathbf{ee}}$ is not the absolute acceleration of the point E but is oppositely directed to it. The absolute acceleration of the point E is the second absolute derivative of the vector $\mathbf{R}_{\mathbf{se}} = \overrightarrow{\mathbf{SE}}$. In this case it is the central acceleration and has the same direction as axis Ez_e. This direction is the correct direction of the gravitational acceleration when it is considered as equivalent central acceleration.

Such gravitational interpretation is a quite new and, among the existing models of the Earth, the model which is, to a certain degree, most appropriate for this interpretation is the Hollow Earth model http://en.wikipedia.org/wiki/ Hollow_Earth.

5. DISTURBED MOTION

A mechanical system is in relative equilibrium when all its points are at rest with respect to the undisturbed sphere Σ. Its relative motion arises when at least one of its points starts move with respect to the sphere Σ. We shall analyze the kinematical quantities that represent relative motion of a point on the mechanical system with respect to the corresponding points on the undisturbed sphere.

Let the motion of the sphere Σ' represent disturbed motion of the sphere Σ. The relative motion of the sphere Σ' with respect to the sphere Σ can be divided into two independent parts. For that purpose let the sphere Σ_1 be introduced; its radius is at all times equal to the radius of the sphere Σ, and the angular velocity is equal to that of the sphere Σ'. During disturbed motion, the point E' moves from point E on the sphere Σ to the point E_1 on the sphere Σ_1 and along the direction $\overline{\mathbf{E_1S}}$, into its final disturbed position E' on the sphere Σ'. It is demanded that the points E_1, E' and S belong to the same direction in order to separate the deviations with respect to undisturbed motion into pure rotation of the sphere and pure deformation of the sphere (Fig. 3 and 4).

The coordinate frame $Ex_ey_ez_e$ will change its orientation and move on to the sphere Σ' into the coordinate frame $E'x'_ey'_ez'_e$ with unit vectors $\underline{\mathbf{e}}_{e'} = (\mathbf{i}_{e'} \quad \mathbf{j}_{e'} \quad \mathbf{k}_{e'})^{\mathrm{T}}$. Let the disturbed motion in the coordinate frame $E'x'_ey'_ez'_e$ (from this point on, all quantities will be expressed in this coordinate frame) be described. The vector that determines the position of the point S with respect to the point E' and the angular velocity of the sphere Σ' are:

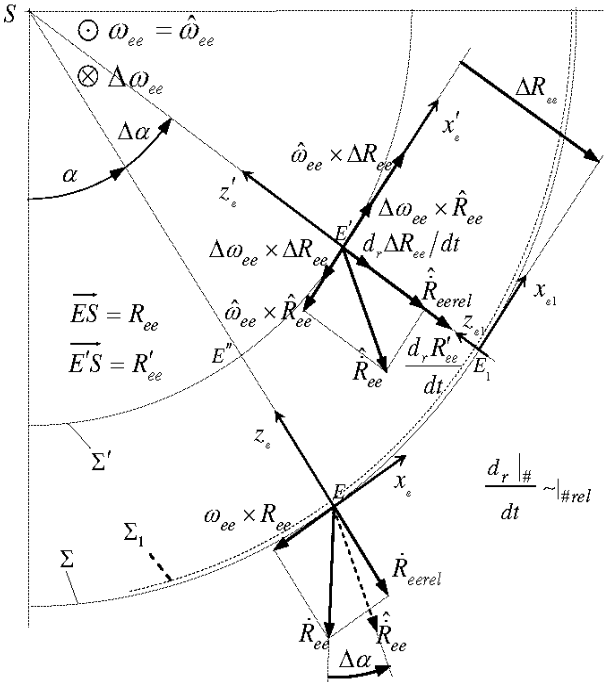

Figure 4: Components of the vector velocity R'_{ee}

$$\mathbf{R}'_{\mathbf{ee}} = \hat{\mathbf{R}}_{\mathbf{ee}} + \Delta\mathbf{R}_{\mathbf{ee}} = (0 \quad 0 \quad R - \Delta R)\underline{\mathbf{e}}_{e'} \quad (4)$$

$$\boldsymbol{\omega}'_{\mathbf{ee}} = \hat{\boldsymbol{\omega}}_{\mathbf{ee}} + \Delta\boldsymbol{\omega}_{\mathbf{ee}} = (\Delta\omega^x_{ee} \quad \dot{\alpha} + \Delta\omega^y_{ee} \quad \Delta\omega^z_{ee})\underline{\mathbf{e}}_{e'} \quad (5)$$

where: $\hat{\mathbf{R}}_{\mathbf{ee}} = \overline{\mathbf{E_1S}} = (0 \quad 0 \quad R)\underline{\mathbf{e}}_{e'}$, $\Delta\mathbf{R}_{\mathbf{ee}} = \overrightarrow{\mathbf{E'E_1}} = (0 \quad 0 \quad -\Delta R)\underline{\mathbf{e}}_{e'}$, and $\hat{\boldsymbol{\omega}}_{\mathbf{ee}} = (0 \quad \dot{\alpha} \quad 0)\underline{\mathbf{e}}_{e'}$, $\Delta\boldsymbol{\omega}_{\mathbf{ee}} = (\Delta\omega^x_{ee} \quad \Delta\omega^y_{ee} \quad \Delta\omega^{\mathbf{z}}_{\mathbf{ee}})\underline{\mathbf{e}}_{e'}$. At increments $\Delta\boldsymbol{\omega}_{\mathbf{ee}}$, coordinate frame $E'x'_ey'_ez'_e$ will change orientation with respect to the coordinate frame $Ex_ey_ez_e$ for angles $\Delta\alpha = \int_{t_0}^{t} \Delta\omega^x_{ee}dt$, $\Delta\beta = \int_{t_0}^{t} \Delta\omega^y_{ee}dt$, $\Delta\gamma = \int_{t_0}^{t} \Delta\omega^z_{ee}dt$. It is supposed that the change in orientation is small. The vector $\hat{\mathbf{R}}_{\mathbf{ee}}$ is the component of the position vector $\mathbf{R}'_{\mathbf{ee}}$ whose projections to the axes of the coordinate frame $E'x'_ey'_ez'_e$ are identical to the projections of the vector $\mathbf{R}_{\mathbf{ee}} = \overrightarrow{\mathbf{ES}}$ to the axes of the coordinate frame $Ex_ey_ez_e$. The same applies for the vector of angular velocity $\hat{\omega}_{\mathbf{ee}}$, and any other vector that is assigned " $\hat{}$ " in the continuation.

The velocity of the position vector $\overrightarrow{\mathbf{E'S}} = \mathbf{R}'_{\mathbf{ee}}$ of the centre of rotation S is

$$\begin{aligned} \dot{\mathbf{R}}'_{\mathbf{ee}} &= \dot{\hat{\mathbf{R}}}_{\mathbf{ee}} + \Delta\dot{\mathbf{R}}_{\mathbf{ee}} = \frac{d_{e'}\mathbf{R}'_{\mathbf{ee}}}{dt} + (\hat{\boldsymbol{\omega}}_{\mathbf{ee}} + \Delta\boldsymbol{\omega}_{\mathbf{ee}}) \times \mathbf{R}'_{\mathbf{ee}} = \\ &= \dot{\hat{\mathbf{R}}}_{\mathbf{ee}} + \Delta\dot{\mathbf{R}}_{\mathbf{ee}}\Big|_{\hat{\boldsymbol{\omega}}_{\mathbf{ee}}=0} + \Delta\dot{\mathbf{R}}_{\mathbf{ee}}\Big|_{\substack{\Delta\boldsymbol{\omega}_{\mathbf{ee}}=0 \\ \Delta R=const.}} \end{aligned} \quad (6)$$

where: $d_e(\cdot)/dt$ denotes relative derivative of a vector with respect to coordinate frame $E'x'_ey'_ez'_e$ and

$$\dot{\hat{\mathbf{R}}}_{\mathbf{ee}} = \dot{\hat{\mathbf{R}}}_{\mathbf{ee}} + \Delta\left(\dot{\hat{\mathbf{R}}}_{\mathbf{ee}}\right),$$

$$\dot{\hat{\mathbf{R}}}_{\mathbf{ee}} = \frac{d_{e'}\hat{\mathbf{R}}_{\mathbf{ee}}}{dt} + \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \hat{\mathbf{R}}_{\mathbf{ee}} = \underline{\mathbf{e}}_{e'}^{\mathrm{T}} \begin{pmatrix} R\dot{\alpha} \\ 0 \\ \dot{R} \end{pmatrix},$$

$$\Delta\left(\dot{\hat{\mathbf{R}}}_{\mathbf{ee}}\right) = \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \hat{\mathbf{R}}_{\mathbf{ee}} = \underline{\mathbf{e}}_{e'}^{\mathrm{T}} \begin{pmatrix} R\Delta\omega_{ee}^{y} \\ -R\Delta\omega_{ee}^{x} \\ 0 \end{pmatrix},$$

$$\Delta\dot{\mathbf{R}}_{\mathbf{ee}}\Big|_{\hat{\omega}_{\mathbf{ee}}=0} = \frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt} + \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \Delta\mathbf{R}_{\mathbf{ee}} = \underline{\mathbf{e}}_{e'}^{\mathrm{T}} \begin{pmatrix} -\Delta R\Delta\omega_{ee}^{y} \\ \Delta R\Delta\omega_{ee}^{x} \\ -\Delta\dot{R} \end{pmatrix},$$

$$\Delta\dot{\mathbf{R}}_{\mathbf{ee}}\Big|_{\substack{\Delta\omega_{\mathbf{ee}}=0 \\ \Delta R=const.}} = \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \Delta\mathbf{R}_{\mathbf{ee}} = \underline{\mathbf{e}}_{e'}^{\mathrm{T}} \begin{pmatrix} -\Delta R\dot{\alpha} \\ 0 \\ 0 \end{pmatrix}.$$

The term $\dot{\hat{\mathbf{R}}}_{\mathbf{ee}}$ represents the velocity of the point E_1 which is the sum of two terms. The first term $\dot{\hat{\mathbf{R}}}_{\mathbf{ee}}$ is the velocity of the point E_1^{Σ} that coincides with the point E_1 but belong to the sphere Σ. The second term $\Delta\left(\dot{\hat{\mathbf{R}}}_{\mathbf{ee}}\right)$ is the velocity of the point E_1 with respect to the E_1^{Σ}. The term $\Delta\dot{\mathbf{R}}_{\mathbf{ee}}\big|_{\hat{\omega}_{\mathbf{ee}}=0}$ is a part of the increment $\Delta\dot{\mathbf{R}}_{\mathbf{ee}}$ of the velocity $\dot{\hat{\mathbf{R}}}_{\mathbf{ee}}$ describing the motion of the point E' around the point E_1 as if the sphere Σ does not rotate, and the sphere Σ' rotate at angular velocity $\Delta\boldsymbol{\omega}_{\mathbf{ee}}$. The term $\Delta\dot{\mathbf{R}}_{\mathbf{ee}}\big|_{\substack{\Delta\omega_{\mathbf{ee}}=0 \\ \Delta R=const.}}$ is a part of the increment $\Delta\dot{\mathbf{R}}_{\mathbf{ee}}$ of the velocity $\dot{\hat{\mathbf{R}}}_{\mathbf{ee}}$ describing the difference of peripheral velocities of the points E_1 and E' when the spheres Σ_1 and Σ' rotates at same angular velocity $\hat{\boldsymbol{\omega}}_{\mathbf{ee}}$. Figure 3 shows all the components of velocity of the vector $\mathbf{R}'_{\mathbf{ee}}$ in case when $\Delta\boldsymbol{\omega}_{\mathbf{ee}} = (0 \quad \Delta\omega_{ee}^{y} \quad 0)\underline{\mathbf{e}}_{e'}$. In this case, the motion of the vector $\mathbf{R}'_{\mathbf{ee}}$ will be planar, and the observed coordinate frame connected at the point E' will not rotate with respect to the vector $\mathbf{R}'_{\mathbf{ee}}$.

The linear acceleration of the origin of the coordinate frame $E'x'_e y'_e z'_e$ is

$$\begin{aligned} \ddot{\mathbf{R}}'_{\mathbf{ee}} &= \ddot{\hat{\mathbf{R}}}_{\mathbf{ee}} + \Delta\ddot{\mathbf{R}}_{\mathbf{ee}} = \frac{d_{e'}\dot{\mathbf{R}}'_{\mathbf{ee}}}{dt} + (\hat{\boldsymbol{\omega}}_{\mathbf{ee}} + \Delta\boldsymbol{\omega}_{\mathbf{ee}}) \times \dot{\mathbf{R}}'_{\mathbf{ee}} = \\ &= \ddot{\hat{\mathbf{R}}}_{\mathbf{ee}} + \Delta\ddot{\mathbf{R}}_{\mathbf{ee}}\Big|_{\hat{\omega}_{\mathbf{ee}}=const.=0} + \Delta\ddot{\mathbf{R}}_{\mathbf{ee}}\Big|_{\substack{\Delta\omega_{\mathbf{ee}}=const.=0 \\ \Delta R=const.}} + \Delta\ddot{\mathbf{R}}_{\mathbf{ee}rez} \end{aligned} \quad (7)$$

where

$$\ddot{\hat{\mathbf{R}}}_{\mathbf{ee}} = \ddot{\hat{\mathbf{R}}}_{\mathbf{ee}} + \Delta\left(\ddot{\hat{\mathbf{R}}}_{\mathbf{ee}}\right)$$

$$\begin{aligned} \ddot{\hat{\mathbf{R}}}_{\mathbf{ee}} &= \frac{d_{e'}\dot{\hat{\mathbf{R}}}_{\mathbf{ee}}}{dt} + \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \dot{\hat{\mathbf{R}}}_{\mathbf{ee}} = \frac{d_{e'}^{2}\hat{\mathbf{R}}_{\mathbf{ee}}}{dt^{2}} + \frac{d_{e'}\hat{\boldsymbol{\omega}}_{\mathbf{ee}}}{dt} \times \hat{\mathbf{R}}_{\mathbf{ee}} + \\ &+ \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times (\hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \hat{\mathbf{R}}_{\mathbf{ee}}) + 2\hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \frac{d_{e'}\hat{\mathbf{R}}_{\mathbf{ee}}}{dt} = \underline{\mathbf{e}}_{e'}^{\mathrm{T}} \begin{pmatrix} \ddot{\alpha}R + 2\dot{\alpha}\dot{R} \\ 0 \\ \ddot{R} - R\dot{\alpha}^{2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \Delta\left(\ddot{\hat{\mathbf{R}}}_{\mathbf{ee}}\right) &= \frac{d_{e'}\Delta\boldsymbol{\omega}_{\mathbf{ee}}}{dt} \times \hat{\mathbf{R}}_{\mathbf{ee}} + \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \frac{d_{e'}\hat{\mathbf{R}}_{\mathbf{ee}}}{dt} + \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \dot{\hat{\mathbf{R}}}_{\mathbf{ee}} + \\ &+ \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \left(\Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \hat{\mathbf{R}}_{\mathbf{ee}}\right) + \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \left(\Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \hat{\mathbf{R}}_{\mathbf{ee}}\right) = \\ &= \underline{\mathbf{e}}_{e'}^{\mathrm{T}} \begin{pmatrix} \Delta\dot{\omega}_{\mathbf{ee}}^{y}R + \Delta\omega_{\mathbf{ee}}^{z}\Delta\omega_{\mathbf{ee}}^{x}R + 2\Delta\omega_{\mathbf{ee}}^{y}\dot{R} \\ -\Delta\dot{\omega}_{\mathbf{ee}}^{x}R + \Delta\omega_{\mathbf{ee}}^{z}\Delta\omega_{\mathbf{ee}}^{y}R - 2\Delta\omega_{\mathbf{ee}}^{x}\dot{R} + \Delta\omega_{\mathbf{ee}}^{z}\dot{\alpha}R \\ -2\Delta\omega_{\mathbf{ee}}^{y}\dot{\alpha}R - \left(\Delta\omega_{\mathbf{ee}}^{x}\right)^{2}R - \left(\Delta\omega_{\mathbf{ee}}^{y}\right)^{2}R \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \Delta\ddot{\mathbf{R}}_{\mathbf{ee}}\Big|_{\hat{\omega}_{\mathbf{ee}}=const.=0} &= \frac{d_{e'}^{2}\Delta\mathbf{R}_{\mathbf{ee}}}{dt^{2}} + \frac{d_{e'}\Delta\boldsymbol{\omega}_{ee}}{dt} \times \Delta\mathbf{R}_{\mathbf{ee}} + \\ &+ \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \left(\Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \Delta\mathbf{R}_{\mathbf{ee}}\right) + 2\Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt} = \\ &= \underline{\mathbf{e}}_{e'}^{\mathrm{T}} \begin{pmatrix} -\Delta\dot{\omega}_{\mathbf{ee}}^{y}\Delta R - \Delta\omega_{\mathbf{ee}}^{z}\Delta\omega_{\mathbf{ee}}^{x}\Delta R - 2\Delta\omega_{\mathbf{ee}}^{y}\dot{R} \\ \Delta\dot{\omega}_{\mathbf{ee}}^{x}\Delta R - \Delta\omega_{\mathbf{ee}}^{z}\Delta\omega_{\mathbf{ee}}^{y}\Delta R + 2\Delta\omega_{\mathbf{ee}}^{x}\Delta\dot{R} \\ -\Delta\ddot{R} + \left(\Delta\omega_{\mathbf{ee}}^{x}\right)^{2}\Delta R + \left(\Delta\omega_{\mathbf{ee}}^{y}\right)^{2}\Delta R \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \Delta\ddot{\mathbf{R}}_{\mathbf{ee}}\Big|_{\substack{\Delta\omega_{\mathbf{ee}}=const.=0 \\ \Delta R=const.}} &= \frac{d_{e'}\hat{\boldsymbol{\omega}}_{ee}}{dt} \times \Delta\mathbf{R}_{\mathbf{ee}} + \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times (\hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \Delta\mathbf{R}_{\mathbf{ee}}) = \\ &= \underline{\mathbf{e}}_{e'}^{\mathrm{T}} \begin{pmatrix} -\Delta R\ddot{\alpha} \\ 0 \\ \Delta R\dot{\alpha}^{2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \Delta\ddot{\mathbf{R}}_{\mathbf{ee}rez} &= \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times (\Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \Delta\mathbf{R}_{\mathbf{ee}}) + \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times (\hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \Delta\mathbf{R}_{\mathbf{ee}}) \\ &+ 2\hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt} = \underline{\mathbf{e}}_{e'}^{\mathrm{T}} \begin{pmatrix} -2\Delta\dot{R}\dot{\alpha} \\ -\Delta\omega_{ee}^{z}\dot{\alpha}\Delta R \\ 2\Delta\omega_{ee}^{y}\dot{\alpha}\Delta R \end{pmatrix} \end{aligned}$$

The term $\ddot{\hat{\mathbf{R}}}_{\mathbf{ee}}$ represents the acceleration of the point E_1 which is the sum of two terms. The first term $\ddot{\hat{\mathbf{R}}}_{\mathbf{ee}}$ is the acceleration of the point E_1^{Σ} that coincides with the point E_1 but belong to the sphere Σ. The second term $\Delta\left(\ddot{\hat{\mathbf{R}}}_{\mathbf{ee}}\right)$ is the acceleration of the point E_1 with respect to the E_1^{Σ}. The term $\Delta\ddot{\mathbf{R}}_{\mathbf{ee}}\big|_{\hat{\omega}_{\mathbf{ee}}=0}$ is a part of the increment $\Delta\ddot{\mathbf{R}}_{\mathbf{ee}}$ of the acceleration $\ddot{\hat{\mathbf{R}}}_{\mathbf{ee}}$ describing the motion of the point E' around the point E_1 as if the sphere Σ does not rotate, and the sphere Σ' rotates at angular velocity $\Delta\boldsymbol{\omega}_{\mathbf{ee}}$. The term $\Delta\ddot{\mathbf{R}}_{\mathbf{ee}}\big|_{\substack{\Delta\omega_{\mathbf{ee}}=0 \\ \Delta R=const.}}$ is a part of the increment $\Delta\ddot{\mathbf{R}}_{\mathbf{ee}}$ describing the difference of accelerations of the points

E_1 and E' when the spheres Σ_1 and Σ' rotate at same angular velocity $\hat{\boldsymbol{\omega}}_{\mathbf{ee}}$. The term $\Delta\ddot{\mathbf{R}}_{\mathbf{ee}rez}$ is a part of the increment $\Delta\ddot{\mathbf{R}}_{\mathbf{ee}}$ describing the influence of the relative motion of the sphere Σ' with respect to the sphere Σ when the sphere Σ rotates at angular velocity $\hat{\boldsymbol{\omega}}_{\mathbf{ee}}$.

6. RELATIONSHIP WITH TRANSLATIONAL MOTION

This section will demonstrate how translational motion of a rigid body in the gravitational field can be represented by three independent sphere Σ' motions, to which point E' is attached. Three independent motions of the sphere Σ' are relative motions with respect to the sphere Σ'_s (Fig. 5), which at current instant matches the sphere Σ' but its angular velocity is $\hat{\boldsymbol{\omega}}_{\mathbf{ee}}$ and velocity of radius $\dot{R}$, and are made of:

- two independent relative rotations of the Σ' sphere with respect to the sphere Σ'_s determined by the increment $\Delta\boldsymbol{\omega}_{\mathbf{ee}}$ of the angular velocity vector $\hat{\boldsymbol{\omega}}_{\mathbf{ee}}$. It will be taken that those two rotations take place with respect to two non co-linear axes perpendicular to $\mathbf{R}'_{\mathbf{ee}}$ and intersect at the point S.

- a deforming motion of the sphere Σ' with respect to the sphere Σ determined by means of increment ΔR of the radius R of the sphere Σ.

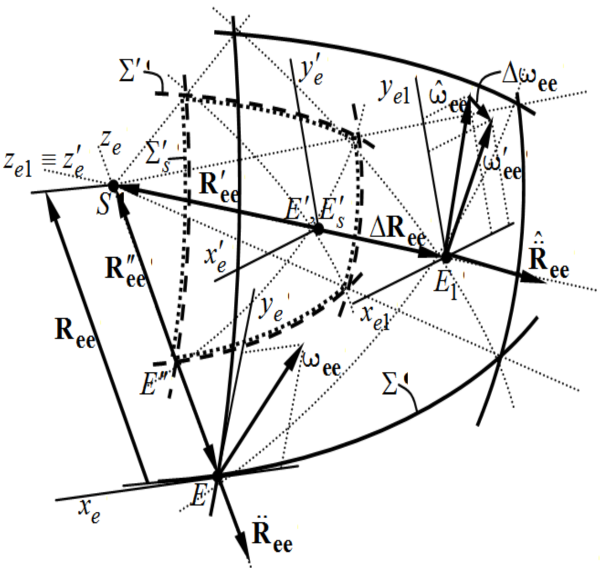

Figure 5: Disturbed and undisturbed motion of the sphere

It will be shown that, at large constant radius of the sphere Σ and additional assumptions about its angular velocity and disturbed motion, translational motion of the rigid body within Earth's gravitational field can be presented as three independent relative sphere motions Σ' with respect to the sphere Σ.

Undisturbed motion of the sphere Σ is given by equations (2) with respect to the referent sphere Σ_0. In vector form, general motion of the point E is defined by the following equations

$$\dot{\mathbf{R}}_{\mathbf{ee}} = \frac{d_e \mathbf{R}_{\mathbf{ee}}}{dt} + \boldsymbol{\omega}_{\mathbf{ee}} \times \mathbf{R}_{\mathbf{ee}} \tag{8}$$

$$\begin{aligned}\ddot{\mathbf{R}}_{\mathbf{ee}} = \frac{d_e^2 \mathbf{R}_{\mathbf{ee}}}{dt^2} + \boldsymbol{\varepsilon}_{\mathbf{ee}} \times \mathbf{R}_{\mathbf{ee}} + \\ + \boldsymbol{\omega}_{\mathbf{ee}} \times (\boldsymbol{\omega}_{\mathbf{ee}} \times \mathbf{R}_{\mathbf{ee}}) + \\ +2\boldsymbol{\omega}_{\mathbf{ee}} \times \frac{d_e \mathbf{R}_{\mathbf{ee}}}{dt}\end{aligned} \tag{9}$$

$$\boldsymbol{\varepsilon}_{\mathbf{ee}} = \frac{d\boldsymbol{\omega}_{\mathbf{ee}}}{dt} = \frac{d_e \boldsymbol{\omega}_{\mathbf{ee}}}{dt} \tag{10}$$

where $\frac{d_e(\cdot)}{dt}$ denotes relative derivative of a vector in the coordinate frame $Ex_e y_e z_e$. If the mass centre of a mass m is at the point E on the sphere Σ, then sphere Σ will act on the mass centre by force in the direction of the acceleration of the point E. Let the reaction force of the sphere be denoted by the $\mathbf{F}_s$ Let $\mathbf{a_E}$ be the acceleration of the mass centre (point E), then, according to the second axiom of the dynamics

$$m\mathbf{a_E} = \mathbf{F_s} \tag{11}$$

and since $\mathbf{a_E} = -\ddot{\mathbf{R}}_{\mathbf{ee}}$ the previous equation receives the form

$$m\ddot{\mathbf{R}}_{\mathbf{ee}} + \mathbf{F_s} = 0 \tag{12}$$

Let the same mass centre be on the Earth's surface. In the state of static balance the following equation must be valid

$$m\mathbf{g} + \mathbf{F_z} = 0 \tag{13}$$

where $\mathbf{g}$ is the acceleration of the gravity, and $\mathbf{F_z}$ the force of reaction of the Earth's surface that acts on the mass centre. The force of reaction $\mathbf{F_z}$ of the Earth's surface is induced by Earth's gravity force $m\mathbf{g}$. If

$$\mathbf{F_z} = \mathbf{F_s} \tag{14}$$

mass centre "will not be aware" if the force acting on it is a result of the Earth's gravity or the fact that it is situated on the sphere Σ and revolves along with it. Let the mass centre be on the sphere Σ. If it is still with respect to the sphere, it will be affected by the reaction force directed towards the centre S. Therefore, the acceleration $\ddot{\mathbf{R}}_{\mathbf{ee}}$ must be on the same course, but in the opposite direction. To fulfill the condition (14) the following must be $|\ddot{\mathbf{R}}_{\mathbf{ee}}| = g$. These conditions within the coordinate frame $Ex_e y_e z_e$ are

$$\ddot{\mathbf{R}}_{\mathbf{ee}} = (0 \quad 0 \quad \ddot{R}_{ee}^z)\underline{\mathbf{e}}_e = (0 \quad 0 \quad -g)\underline{\mathbf{e}}_e \tag{15}$$

Undisturbed motion of the sphere Σ can be any motion for the observed point E where equation (15) holds. The acceleration $\ddot{\mathbf{R}}_{\mathbf{ee}}$ given by relation (17) in the general case of sphere Σ motion in the expanded form in the coordinate frame $Ex_e y_e z_e$ is as follows

$$\ddot{\mathbf{R}}_{\mathbf{ee}} = \underline{\mathbf{e}}_e^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ \ddot{R} \end{pmatrix} + \underline{\mathbf{e}}_e^{\mathrm{T}} \begin{pmatrix} \varepsilon_{ee}^{y_e} \\ -\varepsilon_{ee}^{x_e} \\ 0 \end{pmatrix} R +$$

$$+\underline{\mathbf{e}}_e^{\mathrm{T}} \begin{pmatrix} \omega_{ee}^{z_e}\omega_{ee}^{x_e} \\ \omega_{ee}^{z_e}\omega_{ee}^{y_e} \\ -(\omega_{ee}^{x_e})^2 - (\omega_{ee}^{y_e 2}) \end{pmatrix} R + \underline{\mathbf{e}}_e^{\mathrm{T}} 2 \begin{pmatrix} \omega_{ee}^{y_e} \\ -\omega_{ee}^{x_e} \\ 0 \end{pmatrix} \dot{R} \tag{16}$$

According to equations (15) and (16), undisturbed motion of the sphere Σ is any motion of this sphere in which the following applies

$$\begin{aligned} \dot{\omega}_{ee}^{y_e} R + \omega_{ee}^{z_e}\omega_{ee}^{x_e} R + 2\omega_{ee}^{y_e}\dot{R} &= 0 \\ -\dot{\omega}_{ee}^{x_e} R + \omega_{ee}^{z_e}\omega_{ee}^{y_e} R - 2\omega_{ee}^{x_e}\dot{R} &= 0 \\ \ddot{R} - [(\omega_{ee}^{x_e})^2 + (\omega_{ee}^{y_e})^2]R &= -g \end{aligned} \tag{17}$$

In the equations (17) there are four unknown quantities $\omega_{ee}^{x_e}$, $\omega_{ee}^{y_e}$, $\omega_{ee}^{z_e}$ and R, so in order to solve them an additional condition is required.

The sphere motion is divided into pure deformation and pure rotation so the change in radius R does not depend on the change of the angular velocity $\boldsymbol{\omega}_{\mathbf{ee}}$. From equations (17) the fact that R satisfies the following differential equation is obtained

$$\dddot{R} R + 3\ddot{R}\dot{R} + 3\dot{R}g = 0 \tag{18}$$

Solving it, the following relations are obtained:

$$\ddot{R} = \frac{C_o}{R^3} - g \tag{19}$$

$$\dot{R}^2 = -\frac{C_o}{R^2} - 2gR + C_1 \tag{20}$$

where

$$C_o = R_o^3(\ddot{R}_0 + g), \quad C_1 = \dot{R}_0^2 + R_0\ddot{R}_0 + 3gR_0 \quad \text{and } R_o = R(t_0)$$

Among the all solutions for $\ddot{R}$ and $\dot{R}$ let the following solution be chosen:

$$\begin{aligned} \ddot{R} &= 0 \\ \dot{R} &= 0 \\ \Rightarrow \quad R &= R_0 = const \end{aligned} \tag{21}$$

In that case, with additional condition $\omega_{ee}^{z_e} = 0$, equations (17) are as follows:

$$\begin{aligned} \dot{\omega}_{ee}^{y_e} R_0 &= 0 \\ \dot{\omega}_{ee}^{x_e} R_0 &= 0 \\ [(\omega_{ee}^{x_e})^2 + (\omega_{ee}^{y_e})^2]R_0 &= g \end{aligned} \tag{22}$$

so the solution for the undisturbed motion of the point E is

$$\begin{aligned} \omega_{ee}^{y_e} &= C_{\omega y} = const. \\ \omega_{ee}^{x_e} &= C_{\omega x} = const. \\ \dot{\alpha}^2 R_0 = g, &\quad \dot{\alpha}^2 = C_{\omega x}^2 + C_{\omega x}^2 \end{aligned} \tag{23}$$

Let motion of the sphere Σ' be observed as disturbed motion of the sphere Σ. The velocity of the point E' on the sphere Σ' is given by equations (6). Let point E'' be the intersection of the course ES and sphere Σ_s'. Let its position be determined by the vector $\mathbf{R}_{\mathbf{ee}}'' = \overrightarrow{\mathbf{E}''\mathbf{S}}$. The velocity of the point E' can be expressed as

$$\dot{\mathbf{R}}_{\mathbf{ee}}' = \dot{\hat{\mathbf{R}}}_{\mathbf{ee}}'' = \hat{\dot{\mathbf{R}}}_{\mathbf{ee}}'' + \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \hat{\mathbf{R}}_{\mathbf{ee}}'' \tag{24}$$

where

$$\begin{aligned} \hat{\dot{\mathbf{R}}}_{\mathbf{ee}}'' &= \frac{d_{e'}\hat{\mathbf{R}}_{\mathbf{ee}}''}{dt} + \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \hat{\mathbf{R}}_{\mathbf{ee}}'' = \frac{d_{e'}\hat{\mathbf{R}}_{\mathbf{ee}}}{dt} + \\ &+ \frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt} + \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \hat{\mathbf{R}}_{\mathbf{ee}} + \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \Delta\mathbf{R}_{\mathbf{ee}} = \\ &= \hat{\dot{\mathbf{R}}}_{\mathbf{ee}} + \frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt} + \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \Delta\mathbf{R}_{\mathbf{ee}} \end{aligned} \tag{25}$$

The expression for the velocity $\mathbf{R}_{\mathbf{ee}}'$ takes the form:

$$\dot{\mathbf{R}}_{\mathbf{ee}}' = \quad \hat{\dot{\mathbf{R}}}_{\mathbf{ee}}'' |_{\Delta\dot{R}=0} + \frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt} + \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \mathbf{R}_{\mathbf{ee}}' \tag{26}$$

The kinematical interpretation of this equation is as follows. Observe point E_s' matching the point E', but belonging to sphere Σ'_s (Fig. 5). Then the velocity of the point E_s' is determined by vector $\hat{\dot{\mathbf{R}}}_{\mathbf{ee}}'' |_{\Delta\dot{R}=0}$. The other two elements determine relative velocity of the point E' with respect to point E_s' in three independent directions. The general sense is not diminished by assuming that $\Delta\boldsymbol{\omega}_{\mathbf{ee}}$ is always normal to the direction $\mathbf{R}_{\mathbf{ee}}'$. Then $\frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt}$ in expression (26) represents the relative velocity of the point E' with respect to the point E_s' along the direction SE', and $\Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \mathbf{R}_{\mathbf{ee}}'$ relative velocity in the tangent plain of the sphere Σ'. For instance, the surface of the sphere Σ'_s around the point E' is equivalent to the surface of the sea around a ship sailing on it. At the same time the ship does not have velocity component $\frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt}$, because it is always on the same sea level. If $\Delta\boldsymbol{\omega}_{\mathbf{ee}} = 0$ and $\Delta\dot{R} = 0$ spheres Σ' and Σ'_s do not move with respect to each other and do not change the radius with respect to the sphere Σ. The condition $\Delta\boldsymbol{\omega}_{\mathbf{ee}} = 0$ is insufficient for the spheres Σ' and Σ'_s not to rotate with respect to Σ.

Let analysis of the relations of the intensities of the quantities $\|\Delta\boldsymbol{\omega}_{\mathbf{ee}}\|$ and $\|\Delta\mathbf{R}_{\mathbf{ee}}\|$ and intensities of other quantities be performed. Let it be said that for the motion (30) of the sphere Σ its angular velocity is equal in quantity to the angular velocity of the Earth, i.e.:

$$\begin{aligned} &\| \boldsymbol{\omega}_{\mathbf{ee}} \| = \| \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \| = |\dot{\alpha}| = 2\pi T = \\ &= 2\pi 60 \cdot 60 \cdot 24 \approx \\ &\approx 7.272205217 \cdot 10^{-5} rad/s \end{aligned} \tag{27}$$

then the radius of the sphere Σ will be (according to (23)):

$$R_0 = \frac{g}{\dot{\alpha}^2} = \frac{9.81 \cdot 10^{10}}{7.272205217} \approx 1.854969425 \cdot 10^9 m \tag{28}$$

Momentary peripheral velocity of the sphere Σ is:

$$|\dot{\alpha}| R_0 \approx 1.34897 \cdot 10^5 [m/s] \tag{29}$$

Quantities: sea level, velocity and acceleration of the body with respect to the Earth's surface are more or less within the range of $O^2 = (10^{-1}, 10)^2 = (10^{-2}, 10^2) \cdot$ [corresponding measurement unit]. In this sense it follows:

$$\begin{aligned} &\{\| \Delta\mathbf{R}_{\mathbf{ee}} \|, \left\| \frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt} \right\|, \\ &\| \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \mathbf{R}'_{\mathbf{ee}} \|, g\} \in O^2 \\ &\| \Delta\boldsymbol{\omega}_{\mathbf{ee}} \| \in O^{10} \quad [\ll \inf(O^5)] \\ &|\dot{\alpha}| \in O^5 \quad [\ll \inf(O^2)] \\ &|\dot{\alpha}| R \approx \| \dot{\mathbf{R}}''_{\mathbf{ee}} \| \approx \| \hat{\dot{\mathbf{R}}}''_{ee} |_{\Delta\dot{R}=0} \| \in O^5 \quad [\gg \sup(O^2)] \\ &\| \mathbf{R}'_{\mathbf{ee}} \| \approx \| \mathbf{R}_{\mathbf{ee}} \| \in O^{10} \quad [\gg \sup(O^5)] \end{aligned} \tag{30}$$

The same analysis can also be done for the acceleration of the point E'. Prior to this, it is necessary to express the acceleration of the point E' ($-\ddot{\mathbf{R}}'_{\mathbf{ee}}$) in the following form:

$$\begin{aligned} \ddot{\mathbf{R}}'_{\mathbf{ee}} = {} & \hat{\ddot{\mathbf{R}}}''_{\mathbf{ee}}|_{\Delta\dot{R}=0, \Delta\ddot{R}=0} + \frac{d^2_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt^2} + \Delta\boldsymbol{\varepsilon}_{\mathbf{ee}} \times \hat{\mathbf{R}}''_{\mathbf{ee}} \\ & + \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times (\Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \hat{\mathbf{R}}''_{\mathbf{ee}}) + 2\hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt} \\ & + 2\Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \hat{\dot{\mathbf{R}}}''_{\mathbf{ee}} + (\hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \Delta\boldsymbol{\omega}_{\mathbf{ee}}) \times \hat{\mathbf{R}}''_{\mathbf{ee}} \end{aligned} \tag{31}$$

where:

$$\begin{aligned} \hat{\ddot{\mathbf{R}}}''_{\mathbf{ee}}|_{\Delta\dot{R}=0, \Delta\ddot{R}=0} &= \hat{\ddot{\mathbf{R}}}_{\mathbf{ee}} + \hat{\boldsymbol{\varepsilon}}_{\mathbf{ee}} \times \Delta\mathbf{R}_{\mathbf{ee}} + \hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times (\hat{\boldsymbol{\omega}}_{\mathbf{ee}} \times \Delta\mathbf{R}_{\mathbf{ee}}) \\ \Delta\boldsymbol{\varepsilon}_{\mathbf{ee}} &= \boldsymbol{\varepsilon}'_{\mathbf{ee}} - \hat{\boldsymbol{\varepsilon}}_{\mathbf{ee}} = \\ &= \frac{d_{e'}\Delta\boldsymbol{\omega}_{\mathbf{ee}}}{dt} \\ \hat{\boldsymbol{\varepsilon}}_{\mathbf{ee}} &= \hat{\dot{\boldsymbol{\omega}}}_{\mathbf{ee}} = 0 \end{aligned} \tag{32}$$

Bearing in mind relations (30), some elements of equations (26) and (31) can be neglected, so that the following is obtained:

$$\begin{aligned} \dot{\mathbf{R}}'_{\mathbf{ee}} &\approx \hat{\dot{\mathbf{R}}}_{\mathbf{ee}} + \frac{d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt} + \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \hat{\mathbf{R}}''_{\mathbf{ee}} \\ \ddot{\mathbf{R}}'_{\mathbf{ee}} &\approx \hat{\ddot{\mathbf{R}}}_{\mathbf{ee}} + \frac{d^2_{e'}\Delta\mathbf{R}_{\mathbf{ee}}}{dt^2} + \Delta\boldsymbol{\varepsilon}_{\mathbf{ee}} \times \hat{\mathbf{R}}''_{\mathbf{ee}} \end{aligned} \tag{33}$$

Since, during motion of the point E', the arc $\widehat{EE_1}$ is much smaller than the radius R and since $\hat{\omega}^{z'_e}_{ee} = \Delta\omega^{z'_e}_{ee} = 0$, the coordinate frames $Ex_ey_ez_e$ and $E'x'_ey'_ez'_e$ have approximately the same orientation. Such conditions comply with the translational motion of the coordinate frames $Ex_ey_ez_e$ and $E'x'_ey'_ez'_e$. Then:

$$\begin{aligned} \hat{\dot{\mathbf{R}}}_{\mathbf{ee}} &\approx \dot{\mathbf{R}}_{\mathbf{ee}}, \\ \hat{\ddot{\mathbf{R}}}_{\mathbf{ee}} &\approx \ddot{\mathbf{R}}_{\mathbf{ee}} \end{aligned} \tag{34}$$

In concordance with (21) $d_{e'}\Delta\mathbf{R}_{\mathbf{ee}}/dt = d_{e'}\mathbf{R}'_{\mathbf{ee}}/dt$ holds true. Therefore, the expressions (33) can be written in the form:

$$\begin{aligned} \dot{\mathbf{R}}'_{\mathbf{ee}} &\approx \dot{\mathbf{R}}_{\mathbf{ee}} + \frac{d_{e'}\mathbf{R}'_{\mathbf{ee}}}{dt} + \Delta\boldsymbol{\omega}_{\mathbf{ee}} \times \mathbf{R}'_{\mathbf{ee}} \\ \ddot{\mathbf{R}}'_{\mathbf{ee}} &\approx \ddot{\mathbf{R}}_{\mathbf{ee}} + \frac{d^2_{e'}\mathbf{R}'_{\mathbf{ee}}}{dt^2} + \Delta\boldsymbol{\varepsilon}_{\mathbf{ee}} \times \mathbf{R}'_{\mathbf{ee}} \end{aligned} \tag{35}$$

In the expanded form in the coordinate frame $E'x'_ey'_ez'_e$ the final equations are:

$$\dot{\mathbf{R}}'_{\mathbf{ee}} \approx \mathbf{e}^{\mathrm{T}}_{e'} \begin{pmatrix} C_{\omega y}R \\ -C_{\omega x}R \\ 0 \end{pmatrix} + \mathbf{e}^{\mathrm{T}}_{e'} \begin{pmatrix} \Delta\omega^{y'_e}_{ee}(R - \Delta R) \\ -\Delta\omega^{x'_e}_{ee}(R - \Delta R) \\ -\Delta\dot{R} \end{pmatrix} \tag{36}$$

$$\ddot{\mathbf{R}}'_{\mathbf{ee}} \approx \mathbf{e}^{\mathrm{T}}_{e'} \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} + \mathbf{e}^{\mathrm{T}}_{e'} \begin{pmatrix} \Delta\varepsilon^{y'_e}_{ee}(R - \Delta R) \\ -\Delta\varepsilon^{x'_e}_{ee}(R - \Delta R) \\ -\Delta\ddot{R} \end{pmatrix} \tag{37}$$

The term $(\Delta\omega^{y'_e}_{ee}(R-\Delta R) \quad -\Delta\omega^{x'_e}_{ee}(R-\Delta R) \quad -\Delta\dot{R})\underline{\mathbf{e}}_{e'}$ $\approx (\Delta\omega^{y'_e}_{ee}R \quad -\Delta\omega^{x'_e}_{ee}R \quad -\Delta\dot{R})\underline{\mathbf{e}}_{e'}$ represents the relative velocity of the point E around the point E'. It is approximately equal to the absolute velocity of the point E with respect to the point E', because the its carrying velocity can be neglected. If the point E' belongs to a rigid body, and the point E to the Earth's surface, this element defines translational velocity of the rigid body with respect to the Earth. Analogously, the element

$(\Delta\varepsilon^{y'_e}_{ee}(R - \Delta R) \quad -\Delta\varepsilon^{x'_e}_{ee}(R - \Delta R) \quad -\Delta\ddot{R})\underline{\mathbf{e}}_{e'}$ $\approx (\Delta\varepsilon^{y'_e}_{ee}R \quad -\Delta\varepsilon^{x'_e}_{ee}R \quad -\Delta\ddot{R})\underline{\mathbf{e}}_{e'}$ represents the relative acceleration of the point E with respect to the point E'. It is approximately equal to the absolute acceleration of the point E with respect to the point E', because the carrying and Coriolis acceleration can be neglected. If the point E' belongs to a rigid body and the point E to the Earth's surface, this element defines translational acceleration of the rigid body with respect to the Earth. Therefore, translational motion can be interpreted as partial case of relative motion of the sphere Σ' with respect to sphere Σ.

7. EXPERIMENT FOR VERIFICATION OF KINEMATICAL MODEL OF EARTH MOTION

The assertion that the acceleration of the Earth can be interpreted as a central acceleration is equivalent to the assertion that the motion of the Earth takes place in one of two ways. The first way is that the motion takes place inside the sphere Σ_1 which rotates inside immobile sphere Σ. The central acceleration at the surface of the sphere Σ_1 is equal to the magnitude of Earth's acceleration g. Another way is that the motion of the Earth is same as the motion of objects sliding inside the immobile sphere Σ. Central acceleration is the result of the peripheral sliding velocity, and its magnitude is equal to the magnitude of the Earth's acceleration g.

According to the contemporary theory, Earth is a sphere. Moton of living beings takes place outside the sphere. Motion of the Earth is a complex motion composed of the motion of its mass center along an elliptical orbit and the additional rotation around its axis. On the Earth, gravity acts as an attractive force that attracts objects towards the Earth's surface. This force is equal to $\mathbf{G} = m\boldsymbol{g}$, where m is the mass of an object and $\boldsymbol{g}$ is the Earth's acceleration. The vectors $\mathbf{G}$ and $\boldsymbol{g}$ are directed towards the surface of the Earth, that is, approximately, towards its center.

Relevant data for the Earth are: the period of motion along an elliptical path $365,256$ days, the approximate distance to the Sun $149,600,000 Km$, equatorial radius of the Earth $6,378.14 Km$ and period of rotation $23.9345h$. The central acceleration of the center of the Earth due to the motion along an elliptical trajectory is approximately:

$$a_{sun}^{N} = R_{sun} \cdot \omega_{path}^{2} = 149.6 \cdot 10^{9} * (2\pi / (365.256 \cdot 23.9345 \cdot 60 \cdot 60))^{2} = 5.96 \cdot 10^{-3} [m/s^{2}] = 5.96 [mm/s^{2}].$$

The relative acceleration of a point on Earth with respect to its center is approximately $a_{rel} = R_{earth} \omega_{day}^{2} = 6378140 \cdot (2\pi / 23.9345 \cdot 60 \cdot 60)^{2} = 33.9\ [mm/s^{2}]$. The Coriolis acceleration is approximately:

$$a_{cor} = R_{earth} \omega_{day} \omega_{path} = a_{rel} / 365.256 = 0.1 [mm/s^{2}].$$

Since $R_{earth} \ll R_{sun}$, it can be inferred that the carrying acceleration of any point of the Earth is equal to the carrying acceleration a_{sun}^{N} of the center of the Earth. Also, the direction from any point on the Earth to the Sun approximately coincides with the direction the center of the Earth – the Sun. This direction will be called the Earth – Sun direction. The vector of the central acceleration a_{sun}^{N} is approximately in this direction, directed from the Earth to the Sun.

Choose an arbitrary point on the equator of the Earth and attach firmly to it the origin of a right-handed coordinate frame (Fig. 6). The axis x and y are horizontal. The axis x is oriented to the east, axis y is oriented to the north, and axis z is oriented upward (from the center of the Earth toward its surface).

We shall analyze the acceleration a_x in the direction of the axis x. The axis x is always perpendicular to the direction of g, so the projection of g to the axis x is always zero. The projection of the central acceleration due to the Earth's rotation around its own axis to the axis x is equal to zero, too. The Coriolis acceleration a_{cor} can be neglected.

We shall analyze the projection of the acceleration a_{sun}^{N} due to rotation of the Earth around the Sun to the axis x. Observation starts at noon (at 12 o'clock) when the axis z is approximately directed towards the Sun, i.e. when the center of the Earth, the origin and the Sun are approximately at the same line. The axis x is perpendicular to the Earth - Sun direction at 12 o'clock, so the projection of the acceleration a_{sun}^{N} to the axis x is equal to zero, i.e. $a_x = 0$. At 18 o'clock, Earth turns a quarter of a circle. The axis x becomes approximately collinear with the Earth – Sun direction. The axis x is directed oppositely to the acceleration a_{sun}^{N}, and, the projection of the acceleration to the axis x is equal to $a_x = -|a_{sun}^{N}|$. At 24 o'clock, Earth turns another quarter of a circle. The axis x is again perpendicular to the Earth – Sun direction, i.e. $a_x = 0$. At 6 o'clock Earth turns the $3/4$ of a circle. The axis x again becomes approximately collinear with the Earth – Sun direction. The direction of the axis x and direction of the acceleration a_{sun}^{N} is the same, so the projection of the acceleration to the axis x is equal to $a_x = |a_{sun}^{N}|$. Earth turns entire circle at 12 o'clock and Earth is in the initial position. *The acceleration along the x axis has sinusoidal character with the amplitude of* $|a_x| = |a_{sun}^{N}| = 5.96 [mm/s^{2}]$ *and with the period of* $T_{day} = 23.9345 \approx 24.$ *hour, i.e.* $a_x = -|a_{sun}^{N}| \cdot sin(\omega_{day} t)$, $\omega_{day} = 2\pi / T_{day}$.

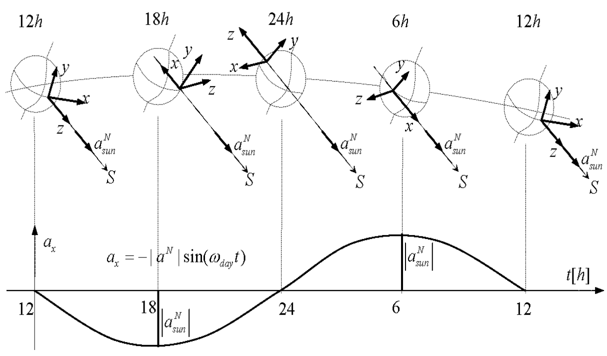

Figure 6: The model of movement of the Earth according to the contemporary science

The difference in the interpretation of gravitational acceleration is reflected primarily in the projection of the acceleration to the direction of the axis x. This difference is simplest to verify experimentally by direct measuring the acceleration along the axis x.

Acceleration of $5.96 \cdot 10^{-3}[m/s^2] = 5.96[mm/s^2]$ is a measurable using accelerometers with a small measurement range. The accelerometer with a range of $\pm 0.1g$ and an accuracy 0.03% of full range output ($0.2g$) can be used [10].

Since $0.2g \cdot 0.03\% = 0.2 \cdot 9.81 \cdot 1000 * 0.0003 = 0.5886[mm/s^2] < 5.96[mm/s^2]$, the character of the acceleration along the axis x can be determined with sufficient accuracy. *The experiment would assume measuring of acceleration during* $24h$ *along horizontal axis* x *directed to the east. If the projection of the acceleration in the direction of the axis* x *is equal to zero, then the Earth does not revolve around its axis, and, contemporary theory about its motion is not correct. If the projection of the acceleration in the direction of the axis* x *has a sinusoidal character with amplitude* $5.96 \cdot 10^{-3}[m/s^2] = 5.96[mm/s^2]$, *then the Earth rotates around its axis, so interpretation of the gravitational acceleration as a central acceleration developed in this paper is not correct.*

8. CONCLUSION

On the basis of exact analysis of known mathematical results for differential geometry of curves, it has been shown that arbitrary motion of a point along a curve can be completely interpreted only by rotation and deformation of a sphere in a kinematical chain of spheres. Spheres of changeable radii in the kinematical chain are positioned in such a fashion that the centre of the next sphere is firmly attached to the previous one within the chain. The kinematical chain of spheres is formed such that the rest of the chain is placed inside the base sphere. The motion of the base sphere has been discussed and in that sense three tasks have presented themselves. The first one is to present the gravitational acceleration as equivalent central acceleration as a consequence of undisturbed base sphere rotation. The undisturbed rotation of the base sphere implies that it is rotating at angular velocity of constant direction with respect to the immobile environment or an environment rotating at a slow rate. The gravitational acceleration, interpreted as a central acceleration, is not directed towards the Earth's surface, but from the Earth's surface. In the second task, the exact and approximate relations for the velocity and acceleration of a point on the base sphere during its disturbed motion have been obtained. The disturbed motion of the base sphere implies rotation and deformation such that there are increments of angular velocity and increments of sphere's radius with respect to angular velocity and sphere's radius during the undisturbed motion. The relations have been presented in the form of the sum of velocity and acceleration of a point on the base sphere during the undisturbed movement and their increments during disturbed motion. Thus, the velocity and acceleration have been expressed as the sum of velocity and acceleration of a pole on the base sphere and rotating velocity and acceleration of the final point in the chain relative to the pole. In the case when the motion takes place at a short distance from the base sphere which is much less than its radius, and when the angular velocity of rotation of the base sphere is negligible, rotating velocity and acceleration can be expressed as function of increments of kinematical quantities that describe disturbed compared to undisturbed motion only. In the third task it has been shown that translational motion in the gravitational field can be treated as a special case of relative motion of disturbed sphere compared to the undisturbed motion of the base sphere. And finally, a model of Earth's motion inside the base sphere has been shown for the case that the gravitational acceleration is equivalent to central acceleration. It has been shown that under such acceleration, objects move towards the base area (i.e towards the surface of the Earth). It has been suggested that the horizontal component of the Earth's acceleration along the axis directed to the east is measured during 24 hours with the precise accelerometers. If the measured acceleration has oscillatory character with a period of 24 hours with amplitude $5.96 \cdot 10^{-3}[m/s^2] = 5.96[mm/s^2]$ then the existing theory of motion of the Earth is correct. If the measured acceleration is constant (equal zero) then the acceleration due to gravity can be interpreted as a central acceleration.

REFERENCES

[1] Vukobratović, M. and Borovac, B.: Zero-Moment Point - Thirty Five Years of its Life, International Journal of Humanoid Robotics, Vol.1, No.1, pp. 157-173, 2004.

[2] Green, A.M., Shaikh, A.G. and Angelaki, D.E.: Sensory vestibular contributions to constructing internal models of self-motion, Journal of Neural Engineering, Vol.2, No.3, pp. 164-179, 2005.

[3] Angelaki, D.E. and Dickmanb, J.D.: Gravity or translation: Central processing of vestibular signals to detect motion or tilt, Journal of Vestibular Research, VES181(11), pp. 1-9, 2002/2003.

[4] Angelaki, D.E., McHenry, M.Q., Dickman, J.D., Newlands, S.D. and Hess, B.J.M.: Computation of Inertial Motion, Neural Strategies to Resolve Ambiguous Otolith Information, The Journal of Neuroscience, Vol.19, No.1, pp. 316-327, 1999.

[5] Živanović, M. and Živanović, Mi. M.: The Increments of Kinematic Quantities of Coordinate System Firmly Attached to Rotating Sphere in *Proc. 50th ETRAN Conference*, pp. 277-281, Belgrade, 2006.

[6] Korn, G. and Korn, T.: *Matematical Handbook for Scientists and Engineers*, Nauka, Moskva, 1984. *(in Russian)*

[7] Živanović, M. and Živanović, Mi. M.: Solid Body Kinematics Expressed in the Natural Trihedron Coordinates in *Proc. 49th ETRAN Conference*, pp. 369-372, Budva, 2005.

[8] Živanović, M. and Živanović, Mi. M.: Dinamically balanced motion of a mechanical systema in *Proc. 51st ETRAN Conference* in RO1.5, pp,1-4, Herceg Novi – Igalo, 2007.

[9] Vinogradov, I. M. and group. *Matematical Ecyclopedia (in Russian).* Sovetskaia Enciklopedia, Moskva, 1977.

[10] AAA320 Extremely rugged, ultra-low range Servo-Accelerometer. Technical report, ALTHEN GmbH Mess- und Sensortechnik, Frankfurter Str. 150 - 152, 65779 Kelkheim, Germany, 2008.

КИНЕМАТИКА БАЗНЕ СФЕРЕ КАО СЕГМЕНТА КИНЕМАТСКОГ ЛАНЦА СФЕРА - ГРАВИТАЦИОНО УБРЗАЊЕ КАО ЕКВИВАЛЕНТНО ЦЕНТРАЛНО УБРЗАЊЕ

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Полазећи од претпоставке да је теорија диференцијалне геометрије тачна, у раду је уведен кинематски ланац сфера промењљивог радијуса постављених тако да је центар сваке суседне сфере чврсто везан за претходну сферу у ланцу. Кинематски ланац се састоји од базне сфере и остатка кинематичког ланца смештеног унутар базне сфере. У раду се разматра непоремећено и поремећено кретање базне сфере. Гравитационо убрзање се интерпретира као еквивалентно централно убрзања настало непоремећеном ротацијом базне сфере. Поремећено кретање у односу на непоремећено кретање се описује помоћу прираштаја одговарајућих кинематских величина. Такође је успостављена веза са важећим законитостима за транслаторно кретање у класичној механици. На крају је предложен и прост експеримент за верификацију теорије предложене у овом раду.