

Further Results on Finite Time and Practical Stability of Linear Continuous Time Delay Systems

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In this paper, finite-time stability and practical stability problems for a class of linear continuous time-delay systems are studied. Based on the Lyapunov-like functions, that do not have to be positive definite in the whole state space and not need to have negative definite derivatives along the system trajectories, the new sufficient finite-time stability conditions are obtained. To obtain the conditions for attractive practical stability, the mentioned approach is combined with classical Lyapunov technique to guarantee attractivity properties of system behavior, and new delay dependent sufficient condition has been derived. The described approach was compared with some previous methods and it has been showed that the results derived are commonly adequate but easier for numerical treatment.

Keywords: linear continuous system, time-delay, finite-time stability, practical stability, Lyapunov function.

1. INTRODUCTION

Classical stability concepts (e.g. Lyapunov stability, BIBO stability) deal with systems operating over an infinite interval of time. These concepts require that system variables be bounded, whereby the values of the bounds are not prescribed. However, often asymptotic stability is not sufficient for practical applications, because there are some cases where large values of the state are not acceptable. In these cases, we need to check that these unacceptable values are not attained by the state. For this purposes, the concepts of the finite-time stability (FTS) and practical stability could be used. A system is said to be FTS if, once a time interval is fixed, its state does not exceed some bounds during this time interval.

Some early results on FTS problems date back to the sixties of the 20th century [1-4]. Recently, the concept of FTS has been revisited in the light of linear matrix inequality theory, which has allowed to find less conservative conditions guaranteeing FTS and finite-time stabilization linear continuous time systems. Many valuable results have been obtained for this type of stability; see, for instance [5-15].

Time delays often occur in many continuous industrial systems (chemical process, biological systems, population dynamics, neural networks, large-scale systems, ...). It has been shown that the existence of delay is the sources of instability and poor performance of control systems. To the best of authors' knowledge, a little work has been done for the finite-time stability and stabilization of continuous time-delay systems. Some early results on finite time

stability of time-delay systems can be found in [16-22]. In [19] and [22] some basic results from the area of finite time stability were extended to the particular class of linear continuous time delay systems using fundamental system matrix. However, these results are not practically applicable, since it requires determining the fundamental system matrix. Matrix measure approach has been, for the first time applied in [17-18, 20] for the analysis of finite time stability of linear time delayed systems. Another approach, based on very well known Bellman-Gronwall Lemma, was applied in [17, 21]. Finally, modified Bellman-Gronwall principle, has been extended to the particular class of continuous non-autonomous time delay systems operating over the finite time interval [16]. The above methods give conservative results because they use boundedness properties of the system response, i.e. of the solution of system models.

Recently, based on linear matrix inequality (LMI) theory, some results have been obtained for FTS and finite-time boundedness (FTB) for some particular class of time-delay systems [23-29]. The papers [23-26] consider the problem of finite-time boundedness (FTB) of the delayed neural networks. The FTS and FTSz of retarded-type functional differential equations are developed in [27]. Papers [28-29] investigate the FTSz problem for networked control systems with time-varying delay. In [29] a particular linear transformation is introduced to convert the original time-delay system into a delay-free form.

The aim of our paper is to present new sufficient conditions of the finite-time stability and attractive practical stability for a class of linear continuous time-delay systems (LCTDS). To solve the problem of FTS we used the Lyapunov-like method. The sufficient delay independent and delay dependent conditions are expressed in the form of algebraic inequality.

Numerical example is used to illustrate the applicability of the developed results

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2. NOTATION AND PRELIMINARIES

\mathbb{R}	Real vector space
\mathbb{C}	Complex vector space
I	Identity matrix
$F \in \mathbb{R}^{n \times n}$	Real matrix
F^T	Transpose of matrix F
$F > 0$	Positive definite matrix
$F \geq 0$	Positive semi definite matrix
$\lambda(F)$	Eigenvalue of matrix F
$\ F\ $	Euclidean matrix norm of F
$\mu(F)$	matrix measure of matrix F

$$\mu(F) = \lim_{\varepsilon \rightarrow 0} \frac{\|1 + \varepsilon F\| - 1}{\varepsilon}$$

The mathematical formulation of the nonlinear differential control systems with time delay can be written:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t-\tau), \mathbf{u}(t)), \quad t \geq 0 \\ \mathbf{x}(t) &= \boldsymbol{\varphi}(t), \quad -\tau \leq t \leq 0 \end{aligned} \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is a state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ is a control vector, $\boldsymbol{\varphi} \in \mathcal{C} = \mathcal{C}([- \tau, 0], \mathbb{R}^n)$ is an admissible initial state function, $\mathcal{C} = \mathcal{C}([- \tau, 0], \mathbb{R}^n)$ is the *Banach* space of continuous functions mapping the interval $[- \tau, 0]$ into \mathbb{R}^n with property of the uniform convergence. The vector function satisfies:

$$\mathbf{f}(\cdot) : \mathfrak{T} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \quad (2)$$

and is assumed to be smooth enough to assure the existence and uniqueness of solutions over a time interval

$$\mathfrak{T} = [t_0, (t_0 + T)] \in \mathbb{R}_+ \quad (3)$$

as well as the continuous dependence of the solutions denoted as $\mathbf{x}(t, t_0, \mathbf{x}_0)$ with respect to t and the initial data. Quantity T is either a positive real number or symbol $+\infty$, so that finite time stability and practical stability can be treated in the same time, respectively.

Under such circumstances, it is not required that

$$\mathbf{f}(t, \mathbf{0}, \mathbf{0}) \equiv \mathbf{0} \quad (4)$$

for an autonomous system, which means that the origin of the state space is not necessarily required to be an equilibrium system state.

Let \mathbb{R}^n denote the state space of system (1) and $\|(\cdot)\|$ Euclidean norm.

Let $V : \mathfrak{T} \times \mathbb{R}^n \rightarrow \mathbb{R}$, be the tentative aggregate function, so that $V(t, \mathbf{x}(t))$ is bounded for and for which $\|\mathbf{x}(t)\|$ is also bounded.

The Eulerian derivative of $V(t, \mathbf{x}(t))$ along the trajectory of system (1), is defined as

$$\begin{aligned} \dot{V}(t, \mathbf{x}(t)) &= \frac{\partial V(t, \mathbf{x}(t))}{\partial t} \\ &+ [\text{grad} V(t, \mathbf{x}(t))]^T \mathbf{f}(\cdot) \end{aligned} \quad (5)$$

For time-invariant sets it is assumed that: $S(\cdot)$ is a bounded, open set.

Let S_β be a given set of all allowable states of the system for $\forall t \in \mathfrak{T}$.

Set S_α , $S_\alpha \subset S_\beta$ denotes the set of all allowable initial states.

Sets S_α , S_β are connected and *a priori* known. $\lambda(\cdot)$ denotes the eigenvalues of a matrix (\cdot) .

λ_{\max} and λ_{\min} are the maximum and minimum eigenvalues, respectively.

Let $\|\mathbf{x}(\cdot)\|_{(\cdot)}$ be any vector norm (e.g., $\cdot = 1, 2, \infty$) and $\|(\cdot)\|$ the matrix norm induced by this vector.

Here, we use $\|\mathbf{x}(t)\|_2 = (\mathbf{x}^T(t) \mathbf{x}(t))^{1/2}$ and $\|A\|_2 = \lambda_{\max}^{1/2}(A^* A)$.

Upper indices * and T denote transpose conjugate and transpose, respectively.

Matrix measure has been widely used in the literature when dealing with stability of time delay systems.

3. MOTIVATION

In our paper we present two quite different approaches to this problem and continue our investigation in a usual way.

Namely, our first result is directly expressed in terms of eigenvalues of basic system matrices A_0 and A_1 naturally occurring in the system model and avoid the need to introduce any canonical form, or transformation into the statement of the theorem.

In the second case the geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions.

This fact makes possible the construction of Lyapunov and Non - Lyapunov stability theory even for the LCTDS in that sense that attractive property is equivalent to the existence of symmetric, positive definite solutions to a general form of Lyapunov matrix equation incorporating condition which refer to boundedness of solutions.

First method is based on a classical approach mostly used in deriving sufficient delay independent conditions for finite time stability.

In the second case a new definition is introduced, based on the attractivity properties of the system solution, which can be treated as something analogous to quasi - contractive stability, [3,4].

Moreover, a quite new delay dependent sufficient condition, has been derived, and it guarantees that the system under consideration will be practically stable with attractive property of its solution, which can be treated as a new concept of so - called non- Lyapunov stability.

4. NECESSARY DEFINITIONS

Let us analyzed a linear continuous stationary system with state delay, described with the following expression:

$$\dot{\mathbf{x}}(t) = A_0 \mathbf{x}(t) + A_1 \mathbf{x}(t - \tau) \quad (6)$$

with a known vector valued function of the initial conditions:

$$\mathbf{x}(t) = \boldsymbol{\varphi}(t), \quad -\tau \leq t \leq 0 \quad (7)$$

where A_0 and A_1 are constant matrices having the appropriate dimensions and τ is constant time delay.

For the stationary system (6) we have: $t_0 = 0$,

$$\mathfrak{T} = [0, T] \text{ and } S_\alpha = \{\mathbf{x}(t) : \mathbf{x}^T(t)\mathbf{x}(t) < \alpha\}.$$

In view of this, we introduce the following definition for finite-time stability and attractive practical stability of time-delay system (6).

Definition 1. System (6) satisfying given initial condition (7) is *finite time stable* w.r.t. to $\{\alpha, \beta, T\}$, where $0 \leq \alpha < \beta$, if

$$\sup_{t \in [-\tau, 0]} \boldsymbol{\varphi}^T(t) \boldsymbol{\varphi}^T(t) \leq \alpha$$

implies

$$\mathbf{x}^T(t) \mathbf{x}(t) < \beta, \quad t \in [0, T].$$

Definition 2. System (6) with initial function (7), is *attractive practically stable* w.r.t. $\{\alpha, \beta, T\}$, if

$$\sup_{t \in [-\tau, 0]} \boldsymbol{\varphi}_x^T(t) \boldsymbol{\varphi}_x(t) \leq \alpha,$$

implies

$$\mathbf{x}^T(t) \mathbf{x}(t) < \beta, \quad t \in [0, T],$$

with property that: $\lim_{t \rightarrow \infty} \mathbf{x}^T(t) \mathbf{x}(t) \rightarrow 0$.

5. FINITE TIME STABILITY

Delay dependent stability conditions

Theorem 1. The autonomous system (6) with the initial function (7) is finite time stable w.r.t. $\{\alpha, \beta, T\}$, if the following condition is satisfied:

$$(1 + \tau \|A_1\|)^2 \|\Phi_0(t)\|^2 < \frac{\beta}{\alpha}, \quad t \in [0, T] \quad (8)$$

where $\Phi_0(t)$ is the fundamental matrix of the system without time delay, see [19].

When $\tau = 0$ or $\|A_1\| = 0$, the problem is reduced to the case of the ordinary linear systems [1].

Theorem 2. The autonomous system (6) with initial function (7) is finite time stable with respect to $\{\alpha, \beta, T\}$ if the following condition is satisfied:

$$(1 + \tau \|A_1\|)^2 e^{2\mu_2(A_0)t} < \frac{\beta}{\alpha}, \quad t \in [0, T] \quad (9)$$

where $\|(\cdot)\|$ denotes the Euclidean norm, [20].

Theorem 3. Time delayed system (6), is finite time stable with respect to $\{\alpha, \beta, T\}$, $\alpha < \beta$, if the following condition is satisfied:

$$e^{\Lambda_{\max} \cdot t} < \frac{\beta}{\alpha}, \quad t \in [0, T] \quad (10)$$

where:

$$\Lambda_{\max} = \Lambda_{1\max} + \tau \cdot \Lambda_{2\max} \quad (11)$$

and:

$$\Lambda_{1\max} = \lambda_{1\max} \left(A_0^T + A_0 + A_1^T + A_1 \right) \\ \Lambda_{\max} = \lambda_{2\max} \left(\begin{array}{c} \wp \left(A_1 A_0 A_0^T A_1^T + A_1 A_1 A_1^T A_1^T \right) \\ + 2 \frac{q^2}{\wp} I \end{array} \right) \quad (12)$$

with: $\wp > 0$ and $q > 1$, [30].

Theorem 4. The autonomous system (6) with initial function (7) is finite time stable with respect to $\{\alpha, \beta, T\}$ if there exist a nonnegative scalar \wp and positive definite symmetric matrices P and Q such that the following conditions hold:

$$\Xi = \begin{pmatrix} A_0^T P + P A_0 + Q - \wp P & P A_1 \\ A_1^T P & -Q \end{pmatrix} < 0 \quad (13)$$

$$\frac{1}{\lambda_{\min}(P)} \left(\lambda_{\max}(P) + \tau \cdot \lambda_{\max}(Q) \right) < \frac{\beta}{\alpha} \quad (14)$$

[31].

In the [31] paper a contemporary procedure, based on linear matrix inequalities (LMI) and Lyapunov - like functions was used to generate sufficient conditions under which the linear time delay system is finite time stable.

Delay independent stability conditions

Theorem 5. System (6) with initial function (7) is finite time stable with respect to $\{\alpha, \beta, T\}$, if the following condition is satisfied:

$$(1 + t \cdot \Sigma_{\max})^2 e^{2t \cdot \Sigma_{\max}} < \frac{\beta}{\alpha}, \quad t \in [0, T] \quad (15)$$

where [17]:

$$\Sigma_{\max} = \sigma_{\max}(A_0) + \sigma_{\max}(A_1). \quad (16)$$

Main results

Theorem 6. System (6) with initial function (7) is finite time stable with respect to $\{\alpha, \beta, T\}$, $\alpha < \beta$, if there exist a positive scalar Λ_{\max} such that the following condition hold:

$$(1 + \tau)e^{\Lambda_{\max} \cdot t} < \frac{\beta}{\alpha}, \quad t \in [0, T] \quad (17)$$

where

$$\begin{aligned} \Lambda_{\max} &= \lambda_{\max}(\Pi), \\ \Pi &= A_0^T + A_0 + A_1 A_1^T + I \end{aligned} \quad (18)$$

with Π being symmetric matrix with all eigenvalues defined over the set of real numbers.

Proof: Let us consider the following Lyapunov-like, aggregation function:

$$\begin{aligned} V(\mathbf{x}(t)) &= \mathbf{x}^T(t) \mathbf{x}(t) + \\ &+ \int_{t-\tau}^t \mathbf{x}^T(\vartheta) \mathbf{x}(\vartheta) d\vartheta \end{aligned} \quad (19)$$

Denote by $\dot{V}(\mathbf{x}(t))$ time derivative of $V(\mathbf{x}(t))$ along the trajectory of system (6), so one can obtain.

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) &= \dot{\mathbf{x}}^T(t) \mathbf{x}(t) + \mathbf{x}^T(t) \dot{\mathbf{x}}(t) \\ &+ \frac{d}{dt} \int_{t-\tau}^t \mathbf{x}^T(\vartheta) \mathbf{x}(\vartheta) d\vartheta \\ &= \mathbf{x}^T(t) (A_0^T + A_0) \mathbf{x}(t) + \\ &+ 2\mathbf{x}^T(t) A_1 \mathbf{x}(t-\tau) + \\ &+ \mathbf{x}^T(t) \mathbf{x}(t) - \mathbf{x}^T(t-\tau) \mathbf{x}(t-\tau) \end{aligned} \quad (20)$$

Based on well-known inequality:

$$\begin{aligned} 2\mathbf{u}^T(t) \mathbf{v}(t-\tau) &\leq \\ &\leq \mathbf{u}^T(t) \Gamma^{-1} \mathbf{u}(t) + \\ &+ \mathbf{v}^T(t-\tau) \Gamma \mathbf{v}(t-\tau), \quad \Gamma > 0 \end{aligned} \quad (21)$$

and with the particular choice $\Gamma = I$, we get:

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) &\leq \\ &\leq \mathbf{x}^T(t) (A_0^T + A_0 + A_1 A_1^T + I) \mathbf{x}(t) \\ &\leq \mathbf{x}^T(t) \Pi \mathbf{x}(t) \leq \\ &\leq \lambda_{\max}(\Pi) \mathbf{x}^T \mathbf{x}(t) \end{aligned} \quad (22)$$

Moreover, if we suppose that $\lambda_{\max}(\Pi) > 0$ it is easy to see:

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) &< \lambda_{\max}(\Pi) \mathbf{x}^T \mathbf{x}(t) \\ &+ \lambda_{\max}(\Pi) \int_{t-\tau}^t \mathbf{x}^T(\vartheta) \mathbf{x}(\vartheta) d\vartheta \\ &< \lambda_{\max}(\Pi) \left(\mathbf{x}^T \mathbf{x}(t) + \int_{t-\tau}^t \mathbf{x}^T(\vartheta) \mathbf{x}(\vartheta) d\vartheta \right) \\ &< \lambda_{\max}(\Pi) V(\mathbf{x}(t)) \end{aligned} \quad (23)$$

since $\int_{t-\tau}^t \mathbf{x}^T(\vartheta) \mathbf{x}(\vartheta) d\vartheta > 0$.

Multiplying (23) with $e^{-\lambda_{\max}(\Pi) \cdot t}$, we can obtain:

$$\frac{d}{dt} \left(e^{-\lambda_{\max}(\Pi) \cdot t} V(\mathbf{x}(t)) \right) < 0 \quad (24)$$

Integrating (24) from 0 to t , with $t \in [0, T]$, we have:

$$V(\mathbf{x}(t)) < e^{\lambda_{\max}(\Pi) \cdot t} \cdot V(0) \quad (25)$$

From (19) it can be seen:

$$\begin{aligned} V(0) &= \mathbf{x}^T(0) \mathbf{x}(0) + \int_{-\tau}^0 \mathbf{\Phi}^T(\vartheta) \mathbf{\Phi}(\vartheta) d\vartheta \\ &\leq \alpha + \alpha \int_{-\tau}^0 d\vartheta \\ &\leq \alpha + \alpha \cdot \tau = \alpha(1 + \tau) \end{aligned} \quad (26)$$

in the light of *Definition 1*.

Combining (25) and (26) leads to:

$$V(\mathbf{x}(t)) < \alpha(1 + \tau) \cdot e^{\Lambda_{\max} \cdot t} \quad (27)$$

On the other hand:

$$\begin{aligned} \mathbf{x}^T(t) \mathbf{x}(t) &< \mathbf{x}^T(t) \mathbf{x}(t) + \int_{t-\tau}^t \mathbf{x}^T(\vartheta) \mathbf{x}(\vartheta) d\vartheta \\ &= V(\mathbf{x}(t)) < \alpha(1 + \tau) \cdot e^{\Lambda_{\max} \cdot t} \end{aligned} \quad (28)$$

Condition (17) and the above inequality imply:

$$\mathbf{x}^T(t) \mathbf{x}(t) < \alpha(1 + \tau) \cdot e^{\Lambda_{\max} \cdot t} < \beta, \quad t \in [0, T] \quad (29)$$

what have to be proved.

In the case of non-delay system, e.g., when $\tau = 0$ or $A_1 = 0$, result given (17) is reduced to [1].

Theorem 7. System (6) with initial function (7) is finite time stable with respect to $\{\alpha, \beta, T\}$, $\alpha < \beta$, if there exists a positive real number q , $q > 1$, such that:

$$\begin{aligned} \|\mathbf{x}(t + \tau)\| &\leq \sup_{\vartheta \in [-\tau, 0]} \|\mathbf{x}(t + \vartheta)\| < q \|\mathbf{x}(t)\|, \\ t &\in [0, T], \quad q > 1, \end{aligned} \quad (30)$$

and if the following condition is satisfied:

$$e^{\bar{\Lambda}_{\max} t} < \frac{\beta}{\alpha}, \quad t \in [0, T] \quad (31)$$

where:

$$\bar{\Lambda}_{\max} = \max \left(\mathbf{x}^T(t) (\Upsilon) \mathbf{x}(t) : \mathbf{x}^T(t) \mathbf{x}(t) = 1 \right) \quad (32)$$

$$\Upsilon = A_0^T + A_0 + A_1 A_1^T + q^2$$

Proof. Define tentative aggregation function as:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t) \mathbf{x}(t) + \int_{t-\tau}^t \mathbf{x}^T(\vartheta) \mathbf{x}(\vartheta) d\vartheta \quad (33)$$

The total derivative $\dot{V}(\mathbf{x}(t))$ along the trajectories of the system, yields:

$$\dot{V}(\mathbf{x}(t)) = \mathbf{x}^T(t) (A_0^T + A_0) \mathbf{x}(t) - \mathbf{x}^T(t-\tau) \mathbf{x}(t-\tau) + 2\mathbf{x}^T(t) A_1 \mathbf{x}(t-\tau) + \mathbf{x}^T(t) \mathbf{x}(t) \quad (34)$$

and based on well-known inequality (21) and with the particular choice $\Gamma = I$, from (34), one can get:

$$\frac{d}{dt} (\mathbf{x}^T(t) \mathbf{x}(t)) \leq \mathbf{x}^T(t) (A_0^T + A_0) \mathbf{x}(t) + \mathbf{x}^T(t) A_1 A_1^T \mathbf{x}(t) + \mathbf{x}^T(t-\tau) I \mathbf{x}(t-\tau) \quad (35)$$

Based on (30), it is clear that (35) reduces to:

$$\frac{d}{dt} (\mathbf{x}^T(t) \mathbf{x}(t)) < \mathbf{x}^T(t) \Upsilon \mathbf{x}(t) \quad (36)$$

where matrix Υ is defined by (32).

From (19) one can get:

$$\frac{d(\mathbf{x}^T(t) \mathbf{x}(t))}{\mathbf{x}^T(t) \mathbf{x}(t)} < \frac{\mathbf{x}^T(t) \Upsilon \mathbf{x}(t)}{\mathbf{x}^T(t) \mathbf{x}(t)} \leq \max \left\{ \frac{\mathbf{x}^T(t) \Upsilon \mathbf{x}(t)}{\mathbf{x}^T(t) \mathbf{x}(t)} \right\} = \max \left(\mathbf{x}^T(t) (\Upsilon) \mathbf{x}(t) : \mathbf{x}^T(t) \mathbf{x}(t) = 1 \right) = \bar{\Lambda}_{\max}$$

or:

$$\int_0^t \frac{d(\mathbf{x}^T(t) \mathbf{x}(t))}{\mathbf{x}^T(t) \mathbf{x}(t)} < \int_0^t \bar{\Lambda}_{\max} dt \quad (38)$$

and:

$$\mathbf{x}^T(t) \mathbf{x}(t) < \mathbf{x}^T(0) \mathbf{x}(0) e^{\bar{\Lambda}_{\max} t} \quad (39)$$

Finally, if one use the first condition of *Definition 3*, then:

$$\mathbf{x}^T(t) \mathbf{x}(t) < \alpha \cdot e^{\bar{\Lambda}_{\max} t} \quad (40)$$

and finally by (31) yields to:

$$\mathbf{x}^T(t) \mathbf{x}(t) < \alpha \cdot \frac{\beta}{\alpha} < \beta, \quad t \in [0, T] \quad (41)$$

what have to be proved.

6. PRACTICAL STABILITY

Delay dependent stability conditions

In this part, the delay dependent stability conditions were analyzed.

Before presenting our crucial result, we need some discussion and explanations as well as some additional results. For the sake of completeness, the following results are presented, as in [32].

Theorem 8. Let the system be described by (6). If for any given positive definite Hermitian matrix Q there exists a positive definite Hermitian matrix P_0 , such that:

$$P_0 (A_0 + P_1(0)) + (A_0 + P_1(0))^* P_0 = -Q \quad (42)$$

where for $\kappa \in [0, \tau]$, $P_1(\kappa)$ satisfies:

$$\dot{P}_1(\kappa) = (A_0 + P_1(0)) P_1(\kappa) \quad (43)$$

with boundary condition $P_1(\tau) = A_1$ and $P_1(\tau) = 0$ elsewhere, then the system is asymptotically stable, [32].

In paper [33] it is emphasized that the key approach in the construction of the Lyapunov function corresponding to system (6) is the existence of at least one solution $P_1(t)$ of (43) with the boundary condition $P_1(\tau) = A_1$. In other words, it is required that the nonlinear algebraic matrix equation

$$e^{(A_0 + P_1(0))\tau} P_1(0) = A_1 \quad (44)$$

has *at least one* solution for $P_1(0)$.

Based on Theorem 7, the asymptotic stability of the system can be determined knowing only one, instead of all solutions of the particular nonlinear matrix equation.

Using counterexample, in [33] is shown that the above theorem is incorrect because it does not take into account all possible solutions of (43), i.e. (44).

Remark 1. If we introduce a new matrix:

$$R \triangleq A_0 + P_1(0) \quad (45)$$

then condition (42) reads:

$$P_0 R + R^* P_0 = -Q \quad (46)$$

which presents a well-known Lyapunov's equation for the system without time delay.

This condition will be fulfilled if and only if R is a stable matrix i.e. if

$$\operatorname{Re} \lambda_i(R) < 0 \quad (47)$$

holds, [33].

Remark 2. Equation (44) expressed through matrix R can be written in a different form as follows:

$$R - A_0 - e^{-R\tau} A_1 = 0 \quad (48)$$

and there follows:

$$\det(R - A_0 - e^{-R\tau} A_1) = 0 \quad (49)$$

[33].

Substituting a matrix variable R by scalar variable s in (47), the characteristic equation of system (6) is obtained as:

$$f(s) = \det(sI - A_0 - e^{-s\tau} A_1) = 0 \quad (50)$$

Let us denote

$$\Sigma = \{s \mid f(s) = 0\} \quad (51)$$

a set of all characteristic roots of system (6), [33].

The necessity for the correctness of the desired results forced us to propose new formulations of *Theorem 8*.

Theorem 9. Suppose that there exist(s) the solution(s) $P_1(0)$ of (44). Then, the system (6) is asymptotically stable *if and only if* any of the two following statements holds:

a) For any matrix $Q = Q^* > 0$ there exists matrix $P_0 = P_0^* > 0$ such that (42) holds for all solutions $P_1(0)$ of (44).

b) The condition $\text{Re } \lambda_i(R) < 0$ holds for all solutions R of (48), [33].

Remark 3. Statements of *Theorem 9* require that the corresponding conditions are fulfilled for any solution $P_1(0)$ of (44) or R of (48).

Remark 4. From the preceding theorems, the following practical question is imposed: how all possible solutions $P_1(0)$ of (44) or R of (48) can be numerically computed? This problem cannot be directly numerically solved because the number of solutions $P_1(0)$ or R is not known beforehand, and can be very large (infinite). However, to more efficiently examine the stability of the system, the above-mentioned numerical problem can be replaced by using maximal solvent of (48) [33].

Definition 3. Each root λ_m of the characteristic equation (50) of the system (6) which satisfies the following condition: $\text{Re } \lambda_m = \max \text{Re } s, s \in \Sigma$ will be referred to as *maximal root (eigenvalue)* of the system (6) [33].

Definition 4. Each solvent R_{\max} of (48), whose spectrum contains maximal eigenvalue λ_m of the system (6), is referred to as *maximal solvent* of (48) [33].

On the basis of *Remark 4* and *Definition 3* and *4*, it is possible to reformulate *Theorem 9* in the following way.

Theorem 10. Suppose that there exists the maximal solvent R_{\max} of (48). Then, system (6) is asymptotically stable *if and only if* any of the two following equivalent statements holds:

a) For any matrix $Q = Q^* > 0$ there exists matrix $P_0 = P_0^* > 0$ such that (46) holds for the solvent R_{\max} .

b) $\text{Re } \lambda_i(R_{\max}) < 0$,

[33] and [34].

Main result

Now, we are in the position to present our main result concerning the practical stability of system (6).

Theorem 11. Autonomous system (6) with initial function (7), is *attractive practically stable* with respect to $\{\alpha, \beta, T\}$, $\alpha < \beta$, if the following condition are satisfied:

$$a) \quad (1 + \tau) e^{\Lambda_{\max} \tau} < \frac{\beta}{\alpha}, \quad t \in [0, T], \quad (52)$$

where

$$\Lambda_{\max} = \lambda_{\max}(\Pi), \quad (53)$$

$$\Pi = A_0^T + A_0 + A_1 A_1^T + I$$

with Π being symmetric matrix with all eigenvalues defined over the set of real numbers.

b) there exists maximal solvent R_{\max} of

$$R - A_0 - e^{-R\tau} A_1 = 0 \quad (54)$$

$$c) \quad \text{Re } \lambda_i(R_{\max}) < 0. \quad (55)$$

Proof. The conditions b) and c) guarantee the asymptotic stability of the system (6). Moreover the finite time stability is provided by (52), what ends the proof.

7. NUMERICAL EXAMPLE

Our main contribution is exposed throughout results of *Theorem 6*.

Example 1. Given a system of the form:

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} -1 & -3 \\ 2 & -3 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0.7 & -1.4 \\ -1.8 & -1.5 \end{pmatrix} \mathbf{x}(t-1)$$

One should investigate *finite time stability* w.r.t. $\{\alpha, \beta, T\}$, with particular choice of: $\alpha = 5.10$, $\beta = 6.0$

and $\boldsymbol{\varphi}(t) = [2 \quad -1]^T$, $-\tau \leq t \leq 0$.

For this purpose, it is suitable to use results of *Theorem 6*.

Note that $\boldsymbol{\varphi}^T(t) \boldsymbol{\varphi}(t) = 5.0 < \alpha$.

Based on (22) the following data can be obtained:

$$\Pi = (A_0^T + A_0 + A_1 A_1^T + I) = \begin{pmatrix} 1.45 & -0.16 \\ -0.16 & 0.49 \end{pmatrix} > 0$$

$$\sigma(\Pi) = \{ \lambda_1 = 0.464, \lambda_2 = 1.476 \}$$

$$\lambda_{\max}(\Pi) = 1.476 > 0$$

$$(1 + \tau) e^{\lambda_{\max}(\Pi) \tau} = (1 + 1) e^{1.476 \cdot (0.15)} \\ = 1.1659 < \frac{\beta}{\alpha} = \frac{6.0}{5.1} = 1.1765$$

so:

$$t = T_{est} = 0.15 [s]$$

Time dependent square norm of systems motion, is shown on Fig. 1.

Overall square norm of systems state trajectory and with more details on Fig. 2, confirming and supporting theoretically results that have been derived. ■

In comparison to real finite time interval $[0, 0.20]_{\beta=6.0}$ it can be concluded that determined estimation is quite good enough, Fig.2

So one can conclude that all conditions of *Theorem 6* are fulfilled, so system under consideration is finite time stable.

To compare our result, let us see check this example by *Theorem 7*.

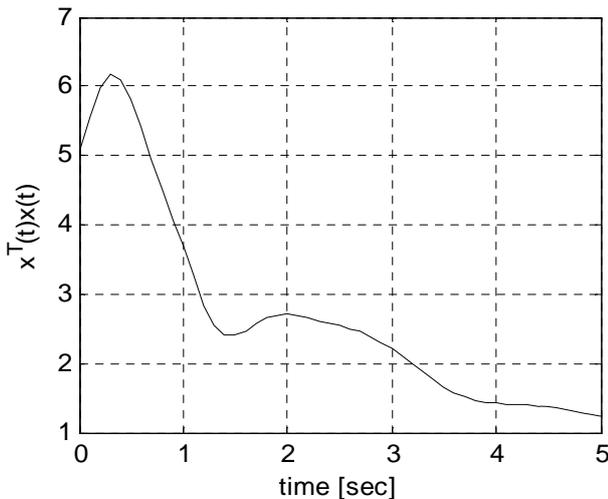


Fig. 1 Time dependent square norm of systems motion.

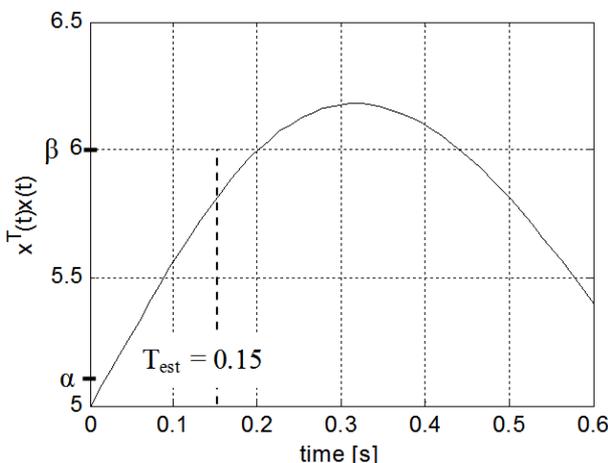


Fig. 2 Detailed square norm of systems state trajectory.

According to (32), matrix Υ is calculated in the following way:

$$\Upsilon = (A_0^T + A_0 + A_1 A_1^T + q^2 I)_{q=1.1} \\ = \begin{pmatrix} 1.6600 & -0.1600 \\ -0.1600 & 0.7000 \end{pmatrix}$$

with:

$$\sigma_{\Upsilon} \{ \lambda_1, \lambda_2 \} = \{ \lambda_1 = 0.6740, \lambda_2 = \lambda_{\max} = 1.6860 \}.$$

Testing (31):

$$e^{\Delta_{\max} \cdot T_{est}} = 1.1639 < \frac{\beta}{\alpha} = 1.1765, T_{est} = 0.09 [s] \quad (56)$$

it can be shown that system is finite time stable over time interval $[0, T_{est}] = [0, 0.09]$ with respect to the given data.

It should be pointed out that (31) is independent delay criteria, while (17) is delay-dependent criteria, which is a certain advantage in any case. Even less restrictive results are derived in our case.

8. CONCLUSION

Generally, this paper extends some of the basic results in the area of the non-Lyapunov stability to linear continuous invariant time-delay systems. Under some certain assumptions the new sufficient, very simple, delay-dependent criteria for the finite time stability of such systems have been presented.

Moreover, a new condition for attractive practical stability is derived combining Lyapunov and non-Lyapunov approach under some circumstances.

Numerical example has been worked out to demonstrate the advantage and efficiency of the methods proposed.

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**НОВИ РЕЗУЛТАТИ У ИЗУЧАВАЊУ
СТАБИЛНОСТИ НА КОНАЧНОМ
ВРЕМЕНСКОМ ИНТЕРВАЛУ И ПРАКТИЧНЕ
СТАБИЛНОСТИ ВРЕМЕНСКИ
НЕПРЕКИДНИХ ЛИНЕАРНИХ СИСТЕМА СА
КАШЊЕЊМ**

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Александра М. Јовановић**

У раду су разматрани проблеми стабилности на коначном временском интервалу и практичне стабилности линеарних временски непрекидних система са кашњењем. Базирајући извођења на квази-Љапуновљевим функцијама, од којих се не захтева позитивна одређеност у целом простору стања, нити негативна одређеност њихових извода дуж кретања система, изведени су нови довољни услови стабилности на коначном временском интервалу. Да би се добили одговарајући услови за атрактивну практичну стабилност, комбинован је претходно поменути прилаз са класичном љапуновском техником, како би се обезбедила особина привлачења кретања система и у том смислу изведени су нови довољни услови зависни од износа чисто временског кашњања. Описани поступак поређен је са неким раније постојећим методама и показано је да су добијају сагласни резултати али далеко једноставнији са становишта нумеричког третмана.