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First Passage of Stochastically Dynamical System with Fractional Derivative and Power-Form Restoring Force Under Gaussian Excitation

First passage problem in a dynamical system with power-form restoring force and fractional derivative is studied in this paper. At first, the original system is transformed into an diffusion differential equation by way of generalized Van der Pol transformation and stochastic averaging method. After that , according to the definition of first passage, BK equation and GP equation are constructed and solved. The numerical results tell us that the reliability probabilities are decreased monotonously with respect to time. Higher order of fractional derivative can lead to higher reliability of the system. Boundary value of safe domain can affect the reliability probability greatly. The larger it is, the higher probability value is.

Keywords: *Fractional derivative, Dynamical system, Restoring force, Gaussian excitation, reliability.*

1. INTRODUCTION

There are a lot of stochastic vibrations in our real life, for example, the tail of aircraft will be strongly vibrated in the air turbulence; the cable of large-scale bridge is vibrated in the strong wind, or high building will vibrate during the earthquake. Generally, we can describe such kind of vibration as stochastic dynamical systems with excitations in mathematics, the air turbulence here or strong wind, earthquake are called random excitations. Since there are stochastic vibrations, the question is, whether the structures can keep safe and stable is that possible to break down during the vibration, how much probability the structures can survive from vibration. If we want to answer these questions, it is needed to study the reliability of stochastic dynamical systems, that is exactly why we want to do this work.

Usually, a stochastic dynamical system is composed of three features, damping, restoring force and random excitation. In the past, most of damping term were modeled as integer-order derivative[1-2], because traditional derivatives have much more good properties and are easy to be dealt with. However, Bagleg[3] and Makris[4] have proved the damping term must be modeled by fractional derivative if the structure is built by some vescoelasity material. Therefore, different from the traditional work, fractional derivative will appear in our systems.

On the other hand, restoring force, in the field of mechanical engineering, is a nonlinear function, as we all know the best nonlinear function is polynomial function, in many research papers, the restoring force is expressed by polynomial functions. For example, in the famous Duffing system, the restoring force is a three-order polynomial function[5]. However, a purely nonlinear function is more reasonable to describe the restoring force, that is why we want to consider the power-form restoring force in our system. As for random excitations, they are all denoted by stochastic processes and satisfy some properties in mathematics. The most popular excitation in mechanical engineering is white noise, bounded noise or color noise [6]. In our paper, we want to take Gaussian white noise into account, the other excitations will appear in the future work.

As a powerful mathematical tool, the fractional derivative has been successfully used in environmental engineering, viscoelasticity material, biomedical science and in vibrated dynamical systems to describe the damping or restoring force[7]. However, the application of fractional derivative in stochastic dynamical system occurred very late, only very few references can be found [8-10].

Therefore, In this paper, we are going to combine the fractional derivative damping, power-form restoring force and Gaussian excitation together, to study first the passage problem in a stochastically dynamical system.

2. MATHEMATICAL MODEL

The motion of equation may be expressed in the form

$$\begin{aligned} &\ddot{x}+\varepsilon f\left(x,\dot{x}\right)D^{\alpha}x\left(t\right)+g\left(x\right) \\ &=\varepsilon^{1/2}h_k\left(x,\dot{x}\right)W_k\left(t\right), \qquad k=1,2,\cdots m \end{aligned} \tag{1}$$

Where $W_k\left(t\right)$ are Gaussian white-noise with zero means and correlation functions $E\left[W_k\left(t\right)W_l(t)\right]=2D_{kl}\delta(\tau)$, D_{kl} is the correlation function, $D^{\alpha}x\left(t\right)$ is Caputo-type fractional derivative governing the damping forces, and defined by

$$D^{\alpha}x\left(t\right)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\dot{x}(t-\tau)}{\tau^{\alpha}}d\tau, \quad 0<\alpha<1 \tag{2}$$

$g(x)$ is a power-form restoring force function and denoted by

Received: June 2014, Accepted: October 2014
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doi:10.5937/fmet1404277L

$$g(x) = c_\beta^2 \operatorname{sgn}(x)|x|^\beta, \tag{3}$$

where β is a non-negative real number.

The objective of this paper is to apply stochastic averaging method to explore the problem of first-passage for the system (1). In doing this, the joint response process $(x, \dot{x})$ is needed to be transformed to a pair of slowly varying processes (a, θ). To that end, a generalized Van der Pol transformation is introduced as follows:

$$x(t) = a(t)\sin[w(a)t + \theta(t)] \tag{4}$$

$$\dot{x}(t) = a(t)w(a)\cos[w(a)t + \theta(t)] \tag{5}$$

Allowing $\varphi(t) = [w(a)t + \theta(t)]$, differentiating Eq.(4) with respect to time and making it equal to Eq. (5), then we get

$$\dot{\theta} = -\frac{\dot{a}\sin\varphi}{a\cos\varphi} \tag{6}$$

and

$$\ddot{x}(t) = \left(\frac{w(a)}{\cos\varphi} + a\cos\varphi\frac{dw(a)}{da}\right)\dot{a} - aw^2(a)\sin\varphi \tag{7}$$

Now consider the nonlinear part of the restoring force in Eq. (1) described by a power-form function. According to the transform in (4), and expanding it into the Fourier series, then we may have

$$g(a,\varphi) = c_\beta^2\left[a^\beta\sum_{n=1}^{\infty} b_{2n-1}(\beta)\sin(2n-1)\varphi\right] \tag{8}$$

where the coefficients are equal to the followings:

$$b_1(\beta) = \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{2+\beta}{2}\right)\Big/\Gamma\left(\frac{3+\beta}{2}\right) \tag{9}$$

$$b_{2n-1}(\beta) = (-1)^n\frac{(\beta-1)(\beta-3)\cdots(\beta-(2n-3))}{(\beta+3)(\beta+5)\cdots(\beta+(2n-1))}b_1(\beta), \quad n = 2,3,\ldots \tag{10}$$

then one expression for frequency of the unforced and un-damping system can be derived from this approximation, which is

$$w^2(a) = c_\beta^2 b_1(\beta)a^{\beta-1}$$

Consequently, a new system in term of $a(t)$ and $\varphi(t)$ can be casted as

$$w(a)\left(\frac{1}{\cos\varphi} + \frac{(\beta-1)\cos\varphi}{2}\right)\dot{a} = -\varepsilon f(a\sin\varphi, aw(a)\cos\varphi)D^\alpha x(t) + \varepsilon^{\frac{1}{2}}h_k(a\sin\varphi, aw\cos\varphi)W_k(t) \tag{11}$$

Apparently, Eq. (11) can be rewriten as follows:

$$\dot{a} = \varepsilon F_1(a,\varphi) + \varepsilon^{1/2}G_{1k}(a,\varphi)W_k(t) \quad k = 1,2,\cdots m \tag{12}$$

where

$$F_1(a,\varphi) = -\frac{\cos\varphi[f(a\sin\varphi, aw\cos\varphi)D^\alpha x(t)]}{w(a)\left[1 + \frac{\beta-1}{2}\cos^2\varphi\right]} \tag{13}$$

$$G_{1k}(a,\varphi) = \frac{\cos\varphi h_k(a\sin\varphi, aw\cos\varphi)}{w(a)\left[1 + \frac{\beta-1}{2}\cos^2\varphi\right]} \tag{14}$$

Substituting (6) into (11), a differential equation that described the phase process is derived

$$\dot{\theta} = \varepsilon F_2(a,\varphi) + \varepsilon^{1/2}G_{2k}(a,\varphi)W_k(t) \tag{15}$$

where

$$F_2(a,\varphi) = \frac{\sin\varphi[f(a\sin\varphi, aw\cos\varphi)D^\alpha x(t)]}{aw(a)\left[1 + \frac{\beta-1}{2}\cos^2\varphi\right]} \tag{16}$$

$$G_{2k}(a,\varphi) = -\frac{\sin\varphi h_k(a\sin\varphi, aw\cos\varphi)}{aw(a)\left[1 + \frac{\beta-1}{2}\cos^2\varphi\right]} \tag{17}$$

According to the stochastic averaging theorem[6], the slowly varying process $a(t)$ converges weakly to an Ito differential equation as $\varepsilon \to 0$. That is

$$da = m(a)dt + \sigma(a)dB(t) \tag{18}$$

where

$$m(a) = \left\langle \varepsilon F_1 + D_{kL}\frac{\partial G_{1k}}{\partial a}G_{1L} + D_{kL}\frac{\partial G_{1k}}{\partial\varphi}G_{2L}\right\rangle_t \tag{19}$$

$$\sigma^2(a) = \langle \varepsilon 2D_{kL}G_{1k}G_{1L}\rangle_t \tag{20}$$

In which $\langle\bullet\rangle_t = \lim_{T\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T \bullet\, dt$

As for the fractional derivative with Caputo definition, combined with formula (4), It will be very useful to introduce the following integrals so as to approximate the Caputo fractional derivative term, they are respectively

$$\int_0^t \frac{\sin w\tau}{\tau^\alpha}d\tau = w^{\alpha-1}\left[\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2} - \frac{\cos wt}{(wt)^\alpha} + o((wt)^{-\alpha})\right] \tag{21}$$

$$\int_0^t \frac{\cos w\tau}{\tau^\alpha}d\tau = w^{\alpha-1}\left[\Gamma(1-\alpha)\sin\frac{\pi\alpha}{2} + \frac{\sin wt}{(wt)^\alpha} + o((wt)^{-\alpha})\right] \tag{22}$$

On the basis of integrals (21) and (22), an approximated function for Caputo-type fractional derivate is obtained.

$$D^\alpha x(t) = aw^\alpha\left(\sin\varphi\cos\frac{\pi\alpha}{2} + \cos\varphi\sin\frac{\pi\alpha}{2}\right) + \frac{aw}{\Gamma(1-\alpha)}\frac{\cos wt - \sin wt}{(wt)^\alpha} + o\left((wt)^\alpha\right) \tag{23}$$

It is evident that the fractional derivative can be approximated by the periodic functions in terms of amplitude and frequency. Substituting formula (23) into (13) and (16), then the drift function and diffusion function will be completedly calculated.

3. RELIABILITY FUNCTION AND MEAN TIME

Based on the differential rule of Ito equation, we can obtain the Markov equation for the total energy function $H(t)$ of the free-vibrated dynamc system (1), which is governed by

$$dH = m(H)dt + \sigma(H)dB(t) \tag{24}$$

shift and diffusion coefficients can be calculated by

$$m(H) = \left[m(a)g(a) + \frac{1}{2}\sigma^2(a)g'(a) \right]_{a=V^{-1}(H)} \tag{25}$$

$$\sigma^2(H) = \left[\sigma^2(a) g^2(a) \right]_{a=V^{-1}(H)} \tag{26}$$

where the potential function $V(a) = \frac{c_\beta^2}{\beta+1} a^{\beta+1}$.

As a matter of fact, the first passage aims to determine the probability that system response reaches the boundary of a randomly excited dynamical system reaches the boundary of a bounded domain of state space within its lifetime. Being one branch of reliability in mathematics, it can exactly describe the response feature and fatigue life of certain structures such as offshore platform, civil construction etc., otherwise the system or the structure will be destroyed once the system crosses beyond the boundary of safe domain. Therefore, conditional reliability function should be :

$$R(t \mid H_0) = p\{H(s) \in D, s \in (0,t) \mid H(0) = H_0 \in D\} \tag{27}$$

where D is the safe domain of energy function. Especially, this reliability function satisfies the following Backward Kolmogorov (BK) equation with conditions

$$\frac{\partial R}{\partial t} = m(H_0)\frac{\partial R}{\partial H_0} + \frac{1}{2}\sigma^2(H_0)\frac{\partial^2 R}{\partial H_0^2} \tag{28}$$

$$R(0 \mid H_0) = 1, \; if \;\; H_0 \in D \tag{29}$$

$$R(t \mid H_0) = 0 \;\; if \;\; H_0 = \partial D \tag{30}$$

$$R(t \mid H_0) \le 1 \;\; if \;\; H_0 = 0 \tag{31}$$

On the other hand, averaged first passage time is another important index to measure the system reliability. We can define the statistical moments of n order as $\mu_n(H_0) = E(T^n)$, and the moments satisfy the following Generalized Pontryagin (GP) equation with conditions:

$$m(H_0)\frac{d\mu_{n+1}}{dH_0} + \frac{1}{2}\sigma^2(H_0)\frac{d^2\mu_{n+1}}{dH_0^2} = -(n+1)\mu_n \tag{32}$$

$$\mu_{k+1}(H_0) = finite, \;\; if \;\; H_0 = 0 \tag{33}$$

$$\mu_{k+1}(H_0) = 0, \;\; if \;\; H_0 = \partial D \tag{34}$$

Obviously, it is also a second order differential equation. Now our objective is not to solve a stochastic dynamical system to get reliability information but to solve these two differential equations (28) and (32) .

4. NUMERICAL RESULTS

The last part will display some reliability results, including the results derived from analytical methods and Monte-Carlo simulation. Consider the following exmple, in which the dynamical system (1) is rewritten as:

$$\ddot{x} + \varepsilon K D^\alpha x(t) + c_\beta^2 \operatorname{sgn}(x)|x|^\beta = \sqrt{\varepsilon} W(t) \tag{35}$$

Figure 1 shows that reliability function is a decreasing function with respect to time. Alpha is the order of fractional derivative, so we can see, more larger alpha values can lead to small higher probability of the system reliability, where we take parameters as follows:

$$\varepsilon = 1, K = 0.05, \beta = 3.5, D_{11} = 0.05, c_\beta = 1, \partial D = 1.0.$$

In this figure, the solid lines are derived from analytical method, but solid squares or triangles are deried from Monte-carlo simulation. They are all in good agreement with each other.

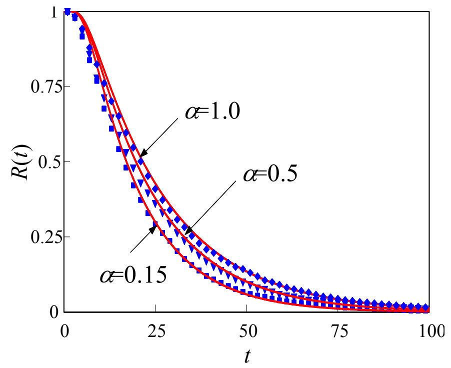

Figure 1. Reliability functions with respect to time, in which $\partial D = 1.0$.

If we increase gradually the boundary value of safe domain, for example $\partial D = 2.0$, then the influence caused by fractional order is more obvious, see Figure 2. Similarly, the results in Figure 2 obtained from analytical method by solving BK eqution and Monte-Carlo simulation from the original system (1) are also in good agreement.

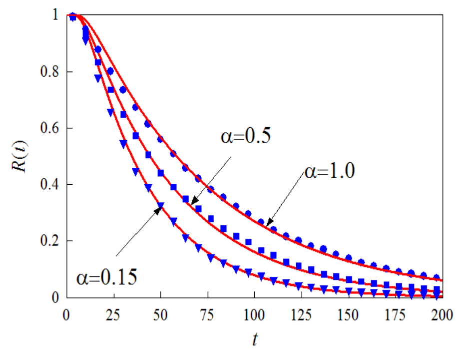

Figure 2. Reliability functions with respect to time, in which $\partial D = 2.0$.

With the same parameters, we also examined the mean first passage time by solving GP equation and Monte-Carlo simulating the original system. Figure 3 shows the relationship among mean first-passage time, fractoinal order of alpha and initial value of energy. If we choose some certain special value of alpha, two-dimension Figure 4 can be more clearly seen the relationship between mean first passage time and initial energy. You can see the results are all consistent with each other. In addition, the higher fractional order, the longer of mean first passage time.

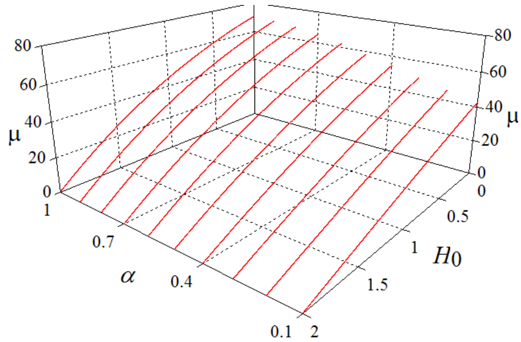

Figure 3. Mean first passage time in three dimension.

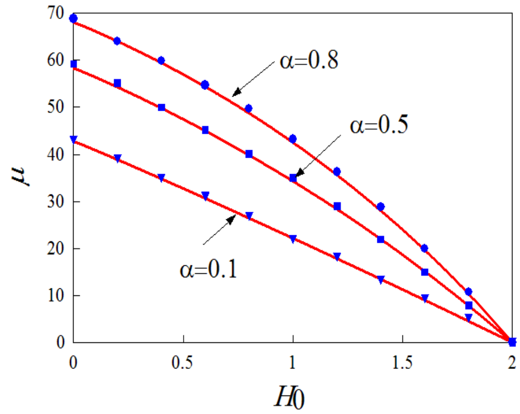

Figure 4. Mean first passage time with respect to initial energy in safe domain.

5. CONCLUSION

To sum up, in this paper, we investigated the first passage problem in a fractional derivative system. Stochastic averaging method has been employed to obtain BK equation and GP equation. The numerical results based on BK equation and GP equation tell us the information about system reliablity, including the effect on reliability by the value of boundary value of safe domain. The comparison between analytical results and Monte-Carlo simulation proved the efficiency and correction of approaches we used in this paper.

ACKNOWLEDGMENT

The work reported in this paper is supported by the National Natural Science Foundation of China (Nos: 11302157, 11202155), "the Fundamental Research Funds for the Central Universities" (No. K5051370008), "Chinese-Serbian Science & Technology Cooperation" (No. 2-14). Also, this research was performed within the TR 35011 and ON 74001, projects supported by Ministry of Science and Technological Development, Republic of Serbia, whose funding is gratefully acknowledged.

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ПРВИ ПРОЛАЗ СТОХАСТИЧКОГ ДИНАМИЧКОГ СИСТЕМА СА ФРАКЦИОНИМ ИЗВОДОМ И СТЕПЕНОМ СИЛОМ ЕЛАСТИЧНОСТИ ПОД ГАУСОВОМ ПОБУДОМ

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У овом раду се разматра Проблем првог пролаза у динамичком систему са степеном силом еластичности и фракционим изводом. Најпре је оригинални систем трансформисан у дифузиону диференцијалну једначину помоћу генералисане Ван дер Полове трансформације и методе стохастичког просека. Након тога захваљујући дефиницији првог пролаза БК једначина и ГП једначина су постављене и решене. Нумерички резултати нам говоре да поузданост вероватноће монотоно опада са временом. Виши редови фракционог извода могу да доведу до веће поузданости система . Граничне вредности сигурне области могу да утичу на поузданост система.