Free Planar Vibration of Structures Composed of Rigid Bodies and Elastic Beam Segments

This article presents free vibration analysis of structures composed of rigid bodies connected with elastic beam segments. It is assumed that the mass centers of rigid bodies are not located on the neutral axes of undeformed elastic beam segments as well as rigid bodies perform planar motion in the same plane and their mass centers are located in that plane. For determination of natural frequencies of the system, modification of the conventional continuous-mass transfer matrix method has been performed. The elastic beam segments are treated as Euler-Bernoulli beams. Numerical example is presented.

Keywords: free vibration analysis, rigid bodies, elastic beams, conventional continuous-mass transfer matrix method, Euler-Bernoulli beams.

1. INTRODUCTION

Many engineering structures can be modeled with a system of rigid bodies connected with elastic beam segments, hence, free vibration analysis of these models of structure are of crucial importance. Many papers deal with vibration analysis of the system composed of a single rigid body and two elastic beam segments [1-3] as well as of the system of cantilever beam with a rigid body attached to its free end [4-6]. In [7] two-dimensional structures composed of two-part elastic beam-rigid body elements are analyzed by using transfer matrix and direct approach. Vibration of hybrid elastic beam carrying several elastic-supported rigid bodies is analyzed in [8]. All above references consider that the mass centers of the rigid bodies are located on the neutral axis of elastic beams.

This paper presents the extension of the existing results of free vibration of structures of rigid bodies connected with elastic beam segments, but unlike existing results, in this paper mass centers of rigid bodies are not located on the neutral axes of elastic beam segments. Also, all elastic beam segments are in the same plane and during oscillations, rigid bodies perform planar motion. For determination of natural frequencies of the system, modification of the conventional continuous-mass transfer matrix method (CTMM) [9] has been performed. Performed modification of CTMM gives the coefficients of lower-order determinant as compared to the determinant obtained in [9], which has importance in numerical analysis of the systems with a large number of elastic beam segments and rigid bodies. Theoretical approach of this paper is based on paper [10]. In this paper, the case when the left side of structure is clamped and the right side of structure is simply supported, is applied.

But the beam is cantilevered and obtained results can be applied easily on any type of constraints on these places.

2. SYSTEM MODELING AND EQUATIONS OF MOTION

A system of rigid bodies \( \alpha_i \) connected by homogenous elastic beam segments (BS) is shown in Fig. 1 [10]. \( C_2 \) represents the mass center of body \( V_2 \). \( \alpha_i \) is the angle between the longitudinal axes of undeformed adjacent segments (BS_1) and (BS_{i+1}). \( O_i \) is the point of body \( V_i \) which represents the intersection point of the longitudinal axes of undeformed adjacent segments (BS_i) and BS_{i+1}. Rigid bodies perform planar motions in the plane where elastic segments are positioned. \( w_i(z_i,t) \) presents the transverse displacement in the \( y_i \) direction and \( u_i(z_i,t) \) presents the axial displacement in the \( z_i \) direction, where \( z_i \) axe coincide with the neutral axis of segment (BS_i).

Figure 1. Structures composed of rigid bodies connected with elastic beams

The partial differential equations for bending and axial vibrations of the beam segments (BS_i) is [11]:

\[
E_i I_i w_i''(z_i,t) + \rho_i A_i \ddot{w}_i(z_i,t) = 0, \quad i = 1, \ldots, n, \quad (1)
\]

\[
\rho_i A_i \ddot{u}_i(z_i,t) - E_i A_i u_i''(z_i,t) = 0, \quad i = 1, \ldots, n, \quad (2)
\]

where \( E_i \) presents modulus of elasticity, \( I(x_i) \) is the cross-sectional area moment of inertia about axis \( x_i \)
which passes through the center of the cross section, \( A_i \) is the cross-sectional area, \( \rho_i \) is the mass density. The beam segments are modeled as the Euler-Bernoulli beams (rotary and shear effects are ignored) [11]. Deformations \( u(z, t) \) and \( w(z, t) \) as well as rotations \( w'(z, t) \) are small.

Using the separation of variables method, the displacements \( w(z, t) \) and \( u(z, t) \) can be written as

\[
w_i(z, t) = W_i(z_i)T(t), \quad u_i(z, t) = U_i(z_i)T(t),
\]

where \( W_i(z) \) and \( U_i(z) \) \((i = 1,...,n)\) are the normal modes in bending and axial vibrations, respectively. According to (3) and (4), (1) and (2) can be rewritten as the following system of \( 2n+1 \) ordinary differential equations:

\[
W_i''(z_i) - k_i^2W_i(z_i) = 0, \quad i = 1,...,n, \tag{5}
\]

\[
U_i''(z_i) + p_i^2U_i(z_i) = 0, \quad i = 1,...,n, \tag{6}
\]

\[
\ddot{T}(t) + \omega^2T(t) = 0, \tag{7}
\]

where \( \omega \) is the natural frequency of vibration of the entire system and

\[
k_i^4 = \frac{\rho_i A_i}{E_i I_i}, \quad p_i^2 = \frac{\rho_i}{E_i}, \quad i = 1,...,n. \tag{8}
\]

The relation between quantities \( k_i \) and \( p_i \) can be seen from (8).

\[
p_i = \sqrt{\frac{I_i(z_i)}{A_i}}k_i^2, \quad i = 1,...,n. \tag{9}
\]

Taking that \( k_i = k \) and \( p_i = \sqrt{I_i(z_i)}A_ik^2 \), from (8) and (9) it follows

\[
k_i = \sqrt{\frac{E_i I_i(z_i) \rho_i A_i}{p_i A_i I_i(z_i)}}k, \quad p_i = \sqrt{\frac{E_i I_i(z_i)}{p_i A_i}}k^2 \tag{10}
\]

and

\[
\omega = \sqrt{\frac{E_i I_i(z_i)}{p_i A_i}}k^2 \tag{11}
\]

The general solutions of (5) and (6) can be expressed as [11]

\[
W_i(z_i) = C_{1i}(\cos(k_i z_i) + C_{2i}(\sin(k_i z_i) + C_{3i}(\sinh(k_i z_i) + C_{4i}(\sinh(k_i z_i)), \tag{12}
\]

\[
U_i(z_i) = C_{5i}(\cos(p_i z_i) + C_{6i}(\sinh(p_i z_i)), \quad i = 1,...,n, \tag{13}
\]

3. BOUNDARY CONDITIONS

3.1 Boundary conditions at the left end beam segment

Let the segment (BS\(_1\)) be clamped at the left end B\(_{1,L}\). Based on this, following boundary conditions hold:

\[
w_1(0, t) = 0, \quad w'_1(0, t) = 0, \quad u_1(0, t) = 0, \tag{14}
\]

which, taking into account (3), (4), (12), (13), (14) can be written in the developed form as follows:

\[
C_{11} + C_{31} = 0, \tag{15}
\]

\[
k_1C_{21} + k_1C_{41} = 0, \tag{16}
\]

\[
C_{51} = 0. \tag{17}
\]

The following matrix relation can be formed:

\[
[C_1] = [C_0] [C_1], \tag{18}
\]

where

\[
[C_0] = \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} & C_{51} & C_{61} \end{bmatrix}, \tag{19}
\]

\[
[C_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{20}
\]

3.2 Boundary conditions of the rigid body (V)

The rigid body (V) is presented in Fig. 2 [10], \( C^*_i \) and \( C^**_i \) represents the perpendicular projections of the mass center \( C_i \) to the directions \( B_{i,R}O_i \) and \( B_{i+1,L}O_i \), respectively.

![Figure 2. Free-body diagram of the body (V)](image-url)

In further considerations the following quantities will be used to describe the material and geometric characteristic of the rigid bodies (V): body mass \( m_{ri} \), mass moment of inertia about centroidal axis \( J_{ri} \),

\[
\bar{O}_{i,R}C_i = e_i, \quad C_i^*B_{i+1,L} = d_i, \quad C_iC_i^* = b_i, \quad \bar{O}_{i,R}B_{i+1,L} = l_i(2). \tag{21}
\]

The slopes of the displacements at the ends \( B_{i,R} \) and \( B_{i+1,L} \) of the segments (BS\(_{i,R}\)) and (BS\(_{i+1}\)) equal the angle of rotation of the body (V):
\[ w_i'(L_i, t) = w_{i+1}'(0, t) \]  \hspace{1cm} (22)

or, in developed form:

\[ k_i \left[ C_{(i)} \sin k_i L_i + C_{2(i)} \cos k_i L_i + \right.
+ \left. C_{(i)} \sinh k_i L_i + C_{4(i)} \cosh k_i L_i \right] =
\]
\[ = k_{i+1} \left( C_{2(i+1)} + C_{4(i+1)} \right). \]  \hspace{1cm} (23)

Further, according to the assumption on small elastic deformations of the beam segments, the displacement vector of point \( O_i \), determined based on the displacement of point \( B_{i+1} \), and the slope \( w_i'(L_i, t) \) reads

\[ (O_i)_{\text{rel}} = \left[ w_i(L_i, t) + B_{i+1} - O_i w_i'(L_i, t) \right] \hat{j}_i + \left. u_i(L_i, t) \right] \hat{k}_i. \]  \hspace{1cm} (24)

Also, the displacement vector of point \( O_i \) can be expressed through the displacement of point \( B_{i+1} \) and deflection \( w_{i+1}'(L_i, t) \) as follows:

\[ (O_i)_{\text{rel}} = \left[ w_{i+1}(0, t) - B_{i+1} + O_i w_{i+1}'(0, t) \right] \hat{j}_i +
+ u_{i+1}(0, t) \hat{k}_i. \]  \hspace{1cm} (25)

Equating (24) and (25) and taking dot product of such obtained expression by the \( \hat{j}_i \) and \( \hat{k}_i \), respectively, yields

\[ u_i(L_i, t) = u_{i+1}(0, t) \cos \alpha_i +
+ \left[ w_{i+1}(0, t) - \ell_i(2) w_{i+1}'(0, t) \right] \sin \alpha_i, \]  \hspace{1cm} (26)

\[ w_i(L_i, t) + \ell_i(2) w_i'(L_i, t) = -u_{i+1}(0, t) \sin \alpha_i +
+ \left[ w_{i+1}(0, t) - \ell_i(2) w_{i+1}'(0, t) \right] \cos \alpha_i, \]  \hspace{1cm} (27)

or in the developed form

\[ C_{5(i)} \cos p_i L_i + C_{6(i)} \sin p_i L_i = C_{5(i+1)} \cos \alpha_i +
+ \left[ C_{1(i+1)} + C_{3(i+1)} \right]
- \ell_i(2) k_{i+1} \left( C_{2(i+1)} + C_{4(i+1)} \right) \sin \alpha_i, \]  \hspace{1cm} (28)

\[ C_{1(i)} \cos k_i L_i + C_{2(i)} \sin k_i L_i + C_{3(i)} \cosh k_i L_i +
+ C_{4(i)} \sinh k_i L_i + \ell_i(1) k_{i+1} \left( -C_{1(i)} \sin k_i L_i +
+ C_{3(i)} \cosh k_i L_i +
+ C_{4(i)} \sinh k_i L_i \right) \]  \hspace{1cm} (29)

\[ = -C_{5(i+1)} \sin \alpha_i + \left[ C_{1(i+1)} +
+ C_{3(i+1)} \right. \left. - \ell_i(2) k_{i+1} \right] \left( C_{2(i+1)} + C_{4(i+1)} \right) \cos \alpha_i. \]

The angular acceleration and the acceleration of the mass center \( C \) of the body (\( V \)), respectively, is

\[ \ddot{e}_i = \ddot{w}_{i+1}(0, t) = -\omega^2 k_{i+1} \left[ C_{2(i+1)} + C_{4(i+1)} \right] \hat{r}(t), \]  \hspace{1cm} (30)

\[ \ddot{a}_C = \ddot{a}_{Bi+1,L} + \ddot{e}_i \times B_{i+1,L} C_i, \]  \hspace{1cm} (31)

where \( \ddot{a}_{Bi+1,L} \) is the acceleration of point \( B_{i+1,L} \), and \( \ddot{e}_i = \ddot{w}_i(L_i, t) \). In (31) on account of assumption about small deformations of the segments, the term \( \ddot{a}_C \times \ddot{a}_B \times B_{i+1,L} C_i \) which represents normal acceleration of the mass center \( C \) is ignored. In that case, \( \ddot{a}_C = \ddot{w}_i(L_i, t) \) is the vector of angular velocity of body (\( V \)). Now, Newton-Euler differential equations of motion of the body is

\[ J_i \dot{\omega}_i = M_{f(i)} - M_{f(i+1)} + F_{M(i)} \dot{e}_i +
+ F_{d(i)} \dot{d}_i + F_{d(i+1)} \dot{d}_i - F_{d(i+1)} b_i, \]  \hspace{1cm} (32)

\[ m_i \left( \ddot{u}_i(0, t) + \dot{b}_i \dot{e}_i \right) =
= F_{d(i+1)} a_i \sin \alpha_i + F_{d(i)} \cos \alpha_i, \]  \hspace{1cm} (33)

\[ m_i \left( \ddot{w}_i(0, t) - a_i \right) \dot{e}_i = F_{d(i+1)} - F_{d(i)} \sin \alpha_i - F_{d(i+1)} \cos \alpha_i. \]  \hspace{1cm} (34)

where \( F_{M(i)} \) and \( F_{d(i+1)} \) are the shear forces of beam segments (\( BS \)) and (\( BS_{i+1} \)), respectively, defined as:

\[ F_{f(i)} = -E_i I_x(i) \ddot{w}_i''(L_i, t), \]  \hspace{1cm} (35)

\[ F_{f(i+1)} = -E_{i+1} I_x(i+1) \ddot{w}_{i+1}''(0, t). \]  \hspace{1cm} (36)

\( F_{a(i)} \) and \( F_{a(d+1)} \) are the axial forces of beam segments (\( BS \)) and (\( BS_{i+1} \)), respectively, defined as:

\[ F_{a(i)} = E_i A_i \dot{u}_i'(L_i, t), \]  \hspace{1cm} (37)

\[ F_{a(i+1)} = E_{i+1} A_{i+1} \dot{u}_{i+1}'(0, t), \]  \hspace{1cm} (38)

and, finally, \( M_{M(i)} \) and \( M_{M(d+1)} \) are the bending moments of beam segments (\( BS \)) and (\( BS_{i+1} \)), respectively, defined as:

\[ M_{f(i)} = -E_i I_x(i) \ddot{w}_i''(L_i, t), \]  \hspace{1cm} (39)

\[ M_{f(i+1)} = -E_{i+1} I_x(i+1) \ddot{w}_{i+1}''(0, t). \]  \hspace{1cm} (40)

Based on above relations, (32)-(34) can be written in a developed form as follows:

\[ -\omega^2 k_{i+1} \left[ C_{2(i+1)} + C_{4(i+1)} \right] =
- E_i I_x(i) k_{i+1} \left[ C_{1(i)} \cos k_i L_i - C_{3(i)} \sin k_i L_i +
+ C_{3(i)} \sinh k_i L_i \right.
+ \left. C_{4(i)} \cosh k_i L_i \right], \]  \hspace{1cm} (41)

\[ - E_{i+1} I_x(i+1) k_{i+1} \left[ C_{1(i+1)} \cos k_{i+1} L_{i+1} - C_{3(i+1)} \sin k_{i+1} L_{i+1} +
+ C_{3(i+1)} \sinh k_{i+1} L_{i+1} \right.
+ \left. C_{4(i+1)} \cosh k_{i+1} L_{i+1} \right]
- E_i A_i \dot{u}_i'(L_i, t) + E_{i+1} A_{i+1} \dot{u}_{i+1}'(0, t), \]  \hspace{1cm} (42)

\[ - m_i \omega^2 \left[ C_{5(i)} + b_i k_{i+1} \left[ C_{2(i+1)} + C_{4(i+1)} \right] \right]
= E_i A_i \dot{u}_i'(L_i, t) - E_{i+1} A_{i+1} \dot{u}_{i+1}'(0, t), \]  \hspace{1cm} (43)

\[ - E_i I_x(i) k_{i+1} \left[ C_{1(i)} \sin k_i L_i - C_{3(i)} \cos k_i L_i +
+ C_{3(i)} \sinh k_i L_i \right]
+ \left. C_{4(i)} \cosh k_i L_i \right] \sin \alpha_i, \]  \hspace{1cm} (44)
Equations (23), (29), (30), (41), (42), and (43) can be written in the matrix form as follows:

\[ T_L C_i = T_R C_{i+1} \quad (44) \]

where \( C_i = [C_{i(0)} C_{i(1)} C_{i(2)} C_{i(3)} C_{i(4)} C_{i(5)} C_{i(6)}] \) is the overall transfer matrix. Finally, based on equations (44), the following recurrence relation can be written as

\[ C_{i+1} = T_i C_i, \quad i = 1, \ldots, n-1. \quad (45) \]

where \( T_i \in \mathbb{R}^{6 \times 6} \) is transfer matrix between the integration constants for beam segments \((BS_i)\). After \( n-1 \) successive application of the recurrence relation (45), it can be obtained:

\[ C_n = T_{n-1} T_{n-2} \cdots T_1 T_0 C_0. \quad (47) \]

### 3.3 Boundary conditions at the right end beam segment

Let the segment \((BS_n)\) be simply supported at the right end \(B_{n,R}\). Based on this, following boundary conditions hold:

\[ w_n(L_n,t) = 0, \quad w_n^r(L_n,t) = 0. \quad (48) \]

which, taking into account equations (3), (4), (12), (13), (48) can be written in the developed form as follows:

\[ C_1(n) \cos(k_n L_n) + C_2(n) \sin(k_n L_n) + \\
C_3(n) \cosh(k_n L_n) + C_4(n) \sinh(k_n L_n) = 0, \quad (49) \]
\[ -k_n^2 C_1(n) \cos(k_n L_n) - k_n^2 C_2(n) \sin(k_n L_n) + \\
k_n^2 C_3(n) \cosh(k_n L_n) + k_n^2 C_4(n) \sinh(k_n L_n) = 0. \quad (50) \]

The following matrix relation can be formed:

\[ T_n C_n = O_{3 \times 1}, \quad (51) \]

where

\[ T_{11(n)} = T_{12(n)} = T_{13(n)} = 0, \quad (52) \]
\[ T_{21(n)} = \cos(k_n L_n), \quad T_{22(n)} = \sin(k_n L_n), \quad (53) \]
\[ T_{23(n)} = \cosh(k_n L_n), \quad T_{24(n)} = \sinh(k_n L_n), \]
\[ T_{25(n)} = T_{26(n)} = 0, \]
\[ T_{31(n)} = -k_n^2 \cos(k_n L_n), \]
\[ T_{32(n)} = -k_n^2 \sin(k_n L_n), \]
\[ T_{33(n)} = k_n^2 \cosh(k_n L_n), \]
\[ T_{34(n)} = k_n^2 \sinh(k_n L_n), \]
\[ T_{35(n)} = T_{36(n)} = 0. \quad (54) \]

### 3.4 Frequency equation and mode shapes

Taking into account (47), it follows from (51) that

\[ TC_0 = O_{3 \times 1} \quad (55) \]

where \( T \in \mathbb{R}^{3 \times 3} \) represents overall transfer matrix determined by the following expression:

\[ T = T_n T_{n-1} \cdots T_1 T_0. \quad (56) \]

Eq. (53) represents a matrix form of the homogeneous system of equations for unknown components of the matrix \( C_0 \). In order that this system can have non-trivial solutions, it is needed to hold that

\[ \det T = 0. \quad (57) \]

### 4. NUMERICAL EXAMPLE

In this example, rigid body with two elastic beam segments is considered (Fig. 3). The beam segments have circular cross section and the rigid body has square cross section. The following values of the system are used: Young’s modulus \( E_1 = E_2 = 2.069 \times 10^1 \text{N/mm}^2 \), mass density \( \rho_1 = \rho_2 = 7500 \text{ kg/m}^3 \), diameters of the beam segments \( D_1 = D_2 = 0.05 \text{ m} \), length of the beam segments \( L_1 = L_2 = 1 \text{ m} \), mass of the rigid body \( m = 50 \text{ kg} \), dimension of the rigid body \( a = 0.5 \text{ m} \).

The first four mode shapes are presented in Figures 4, 5, 6, 7. Figure 8 shows the effect of angle \( \alpha \) on the first four coefficients \( k \). The characteristic equation for angle \( \alpha = \pi/4 \) is presented in Figures 9, 10, 11, 12. Determined coefficients \( k \) from these figures, as well as first four lowest natural frequencies \( \omega \) are presented in table 1.
Figure 5. The second mode shape

Figure 6. The third mode shape

Figure 7. The fourth mode shape

Figure 8. The effect of angle $\alpha$ on the coefficients $k$

Figure 9. Characteristic equation (determination of $k_1$)

Figure 10. Characteristic equation (determination of $k_2$)

Figure 11. Characteristic equation (determination of $k_3$)

Figure 12. Characteristic equation (determination of $k_4$)

Table 1. The first four natural frequency of the system

<table>
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<tr>
<th>Mode</th>
<th>$k$</th>
<th>$\omega$ [rad/s]</th>
</tr>
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<tr>
<td>1</td>
<td>1.24</td>
<td>89.46</td>
</tr>
<tr>
<td>2</td>
<td>3.16</td>
<td>580.96</td>
</tr>
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<td>3</td>
<td>4.57</td>
<td>1215.08</td>
</tr>
<tr>
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<td>3032.83</td>
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5. CONCLUSION

Free vibrations of structures composed of rigid bodies connected with elastic beam segments are presented in this paper. It is assumed that mass centres of rigid bodies are not located on the neutral axes of elastic beam segments. For determination of natural frequencies of the system, modification of the conventional continuous-mass transfer matrix method (CTMM) [9] has been performed. The matrix $T$ can be formed by using software tools like MatLab and Mathematica. Also, using the procedure developed in this paper, with the help of software tools, it can be found easily the solution of equation $det T = 0$ in the analytical form. This provides possibility to analyze dependence on frequencies of any parameter of a given system. Numerical example is provided in order to represent possibilities of the developed procedure.

REFERENCES


АНАЛИЗА СЛОБОДНИХ РАВАНСКИХ ОСЦИЛАЦИЈА СТРУКТУРА САСТАВЉЕНИХ ОД КРУТИХ ТЕЛА И ЕЛАСТИЧНИХ ГРЕДНИХ СЕГМЕНТА

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Овај рад представља анализу слободних вибрација структура састављених од крутих тела међусобно спојених са еластичним гредама. Претпоставља се да се центри маса крутих тела не налазе на неутралној оси недеформисаног еластичног гредног сегмента као и да крута тела врше равно кретање у истој равни и да се њихови центри маса налазе у тој истој равни. За одређивање фреквенција система, модификација класичне "CTMM" методе је употребљена. Еластични гredni сегменти се третирају као Ојлер-Бернулијеве греде. Приказан је нумерички пример.