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Specific Cost Ratio in a Port Modelling by $M/E_k/1$ Queue

The notion of specific cost ratio involves different type of costs of a ship and a port modeled as a queueing system. Using the known general formula for the specific cost ratio of arbitrary port queueing system (denoted as R), here we derive the related expression for R of the $M/E_k/1$ queue, where E_k ($k = 2,3,...$) is the Erlang-k probability distribution. This expression allows us to obtain a theoretical result which can be applied for determining the optimal values of shape parameter k of E_k under given constraints on other performances of the considered port queueing system. The related numerical and the graphical results are also presented. The obtained results would be a useful tool in future research in related subject areas.

Keywords: *Port, $M/E_k/1$ queue, Erlang-k distribution, Specific cost ratio, Traffic intensity.*

1. INTRODUCTION

Recall that the total daily cost of a port queueing system with a certain number of berths (servers) is defined as a sum of total service cost per day, the total ship cost per day and the total marginal cost per day. Then *the specific cost ratio*, usually denoted as R, is defined as a ratio of total daily cost of a port queueing system and the average daily cost of a ship in port. Notice that the optimal numbers of berths and the associated optimal intervals (the so-called the ranges of optimal server capacities; see, e.g., [1]) in the sense of minimization of R for certain single arrival and bulk arrival queueing systems were firstly studied by Noritake [2], Noritake and Kimura [3] and Radmilović et al. [1]. Furthermore, the total cost for different port systems is extensively studied by many authors in the last ten years ([4]-[14]). In particular, this concerns the problem of minimization of the specific cost ratio which is rarely investigated in the earlier literature on the subject. In addition, some queues are used to determine the utilization rate of the process [15], analytical calculation of throughput of the system [16] and to describe the loading processes at seaside link of seaport automobile terminal [17].

The remainder of the paper is organized as follows. In Section 2, using the general formula for the specific cost ratio of any port queueing system, here also denoted by R, we derive the expression for R related to the $M/E_k/1$ queue with one berth (i.e., with one server) and the infinite capacity, where E_k ($k = 2,3,...$) is the Erlang - k probability distribution, which can be used to model service times with a low coefficient of variation (less than one).
As an application of the deduced expression for R, in Section 3 we prove (Proposition) that for any fixed value of traffic intensity θ, the values $R(k, \theta)$ of specific cost ratio of $M/E_k/1$ decrease as k increases.

Applying the analytical results from Sections 2 and 3, some numerical and graphical results are given in Section 4. These results relate to the comparison of the values of specific cost ratio for port queueing systems $M/E_k/1$ with corresponding values of the shape para– meter and the traffic intensity in the ranges $2 \le k \le 9$ and $0.25 \le \theta \le 0.85$, respectively. Concluding remarks are given in Section 5.

2. THE NOTION OF SPECIFIC COST RATIO AND THE RELATED FORMULA FOR M/EK/1 QUEUE

Let us notice that (see, e.g., [18] and [19]) for any given positive integer k = 2,3,… we say that the random variable E_k has an Erlang-k distribution with mean k/μ if E_k is the sum of k independent random variables $X_1, X_2, \ldots, X_k$ having a common exponential distribution with mean $1/\mu$ and the variance σ^2 given by

$$\sigma^2 = \frac{1}{k\mu^2}. \quad (1)$$

In this paper, we consider a port queueing model described as a single-server queue $M/E_k/1$ with one berth (i.e., one server), where ships (customers) arrive according to a Poisson process with rate λ, they are treated in order of arrival and the service times follow the Erlang-k distribution E_k with a shape parameter k = 2,3,4… and a mean k/μ. As noticed above, the Erlang distribution can be used to model service times with a low coefficient of variation (less than one), but it can also arise naturally (see, e.g., [18]). For instance, if a job has to pass, stage by stage, through a series of k independent production stages, where each stage takes a exponentially distributed time. Recall that the study of the $M/E_k/1$ queue is similar to those of the well known $M/E_k/1$ queue. For stability, it is assumed that the *utilization factor* $\rho = \lambda/\mu$ is less than one, and it is equal to the related traffic intensity θ. On the other hand, $M/E_k/1$ queueing model is just the special case of the $M/G/1$ queueing model. Then by the well known *Pollaczek–Khintchine formula* (see, [20] - Subsection

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9.2, p. 348; cf. [12]), the average number of ships (customers) waiting in the queue, L_q, is equal to

$$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}. \quad (2)$$

whence taking $\rho = \theta$ we obtain

$$L_q = \frac{\lambda^2\sigma^2 + \theta^2}{2(1-\theta)}. \quad (3)$$

Then substituting (1) and $\lambda/\mu = \rho$ into (3), we immediately obtain the following formula for the average number of ships (customers) waiting in queue, $L_q(k)$, concerning the port queueing model $M/E_k/1$:

$$L_q(k) = \frac{k+1}{2k}\cdot\frac{\lambda^2}{\mu(\mu-\lambda)}. \quad (4)$$

Finally, replacing $\mu = \lambda/\theta$ into (4), we get

$$L_q(k) = \frac{k+1}{2k}\cdot\frac{\theta^2}{1-\theta}. \quad (5)$$

Note that for $k = 1$ the expression (5) reduces to the well known formula for the average number of customers waiting in the queue $M/M/1$ with the traffic intensity $\theta = \lambda/\mu$ [12].

Recall that the total cost of a port queueing system for considered period consists of a cost related to the berths and a cost related to the ships ([2-4]). Then the associated specific cost ratio, R, is defined as as a ratio of total daily cost of a port modeled as a queueing system and the average daily cost of a ship in port. This definition and some related results can be found in [2-4, 8, 10], and [11] (also see [5, 10, 13]). Following [10] (also see [13]), using a standard calculus, with the notation $c_b/c_s = r$ (the quotient of the average daily cost of a berth, c_b, and the average daily cost of a ship in port, c_s), we find that

$$R = rn_b + \left(L_q + \frac{2\lambda}{\mu}\right) + \lambda\frac{dL_q}{d\lambda} \quad (6)$$

where λ denotes the average number of ships arriving in a port, n_b is a number of berths, L_q is the average number of ships waiting in considered queue system and μ is the average service rate (i.e., $1/\mu$ is the mean service time).

Now we return to the previously described port queueing system modeled as the $M/E_k/1$ queue with a shape parameter $k \in \{2,3,...\}$. Then since $\theta = \lambda/\mu$, we have

$$\frac{dL_q(k)}{d\lambda} = \frac{dL_q(k)}{d\theta}\cdot\frac{d\theta}{d\lambda} = \frac{dL_q(k)}{d\theta}\cdot\frac{1}{\mu}$$

which together with equality $\lambda/\mu = \theta$ and $n_b = 1$ substituting into (6), immediately gives

$$R = r + L_q(k) + 2\theta + \theta\frac{dL_q(k)}{d\theta}. \quad (7)$$

Finally, differentiating the expression (5) with respect to θ and substituting this into (7) with $R(k,\theta)$ instead of R, after a routine calculation gives

$$R(k,\theta) = r + \frac{\theta(4k + 3\theta - 5k\theta - 2\theta^2 + 2k\theta^2)}{2k(1-\theta)^2}. \quad (8)$$

For several port systems it can be of interest to minimize the specific cost ratio (see [4, 7-11]). In particular, the expression (8) allows us to consider related problems for the queue port models $M/E_k/1$ with different values k □ {2,3,...}. Accordingly, some related theoretical and numerical results are given in Sections 3 and 4, respectively.

3. SOME RESULTS FOR SPECIFIC COST RATIO CONCERNING $M/E_k/1$ QUEUE

Now we consider the port queueing system from Section 2 modeled as the $M/E_k/1$ queue with a shape parameter k □ {2,3,4...} and the traffic intensity θ □ (0,1). Motivated by some numerical results from Section 3 of [5], here we prove the following result which indicates an analytical method for determining (in the sense of minimization of related values of $R(k,\theta)$ under some additional conditions) an optimal value of shape parameter k of the Erlang-k distribution E_k involving in the corresponding port queueing system $M/E_k/1$. This result is given as follows.

Proposition. *Let $r > 0$ and let $\theta \in (0,1)$ be any fixed real numbers and let $R(k,\theta)$ be defined by (8). Then $\{R(k,\theta)\}_{k=2}^{\infty}$ is a decreasing sequence. In other words, it holds that*

$$R(2,\theta) > R(3,\theta) > \cdots > R(k,\theta) > R(k+1,\theta) > \cdots$$

for all $r > 0$ and θ □ (0,1).

Proof of Proposition. Since

$$4k + 3\theta - 5k\theta - 2\theta^2 + 2k\theta^2$$
$$= k(4 - 5\theta + 2\theta^2) + \theta(3 - 2\theta),$$

taking this into the expression (8), immediately gives

$$R(k,\theta) = r + \frac{\theta(4 - 5\theta + 2\theta^2)}{2(1-\theta)^2} + \frac{\theta(3-2\theta)}{2k(1-\theta)^2}.$$

Notice that the first two terms on the right hand side of the above equality does not depend on k and the third term of this side decreases as k increases (since $\theta(3-2\theta)/(2(1-\theta)^2) > 0$ for all $\theta \in (0,1)$. This yields the assertion of Proposition. □

In terms of port queueing systems, the above proposition can be reformulated as follows.

Theorem. *Let $M/E_k/1$, $k = 2,3,\ldots$ be the sequence of single-server port queueing systems which have the same values of θ (the traffic intensity) and $r = c_b/c_s$ (the quotient of the average daily cost of a berth, c_b, and the average daily cost of a ship in port, c_s. Then the values $R(k,\theta)$ of specific cost ratio of the queue $M/E_k/1$ decrease as k increases.*

In particular, the above theorem yields that among all the port queueing systems $M/E_k/1$, $K = 2,3,\ldots$ described in this theorem, the maximal value of specific cost ratio is attained for the $M/E_2/1$ queue. Notice that the assertion of the above theorem is illustrated by some numerical and graphical results presented in the following section.

4. NUMERICAL RESULTS

Applying the expression (8), here we give some numerical and graphical results concerning the specific cost ratio for certain port queueing models described as a single-server queue $M/E_k/1$. More precisely, for any given (fixed) value $r > 0$ (the first term on the right hand side of (8)), we give some numerical and graphical results for the difference $R - r$ (which does not depend on r), where R is given by (8), as a function of the shape parameter k (k = 2,3,4,…,9) and the traffic intensity θ ($0.25 \le \theta \le 0.85$). For these purposes, we write the expression (8) as a sum

$$R(k,\theta) = r + f(k,\theta),$$

where

$$f(k,\theta) = \frac{\theta(4k + 3\theta - 5k\theta - 2\theta^2 + 2k\theta^2)}{2k(1-\theta)^2}. \qquad (9)$$

The graphic of the above function $f(k,\theta)$ in two variables $k \in \{2,3,4,\ldots,9\}$ and $\theta \in [0.2,0.8]$ is presented in Figure 1.

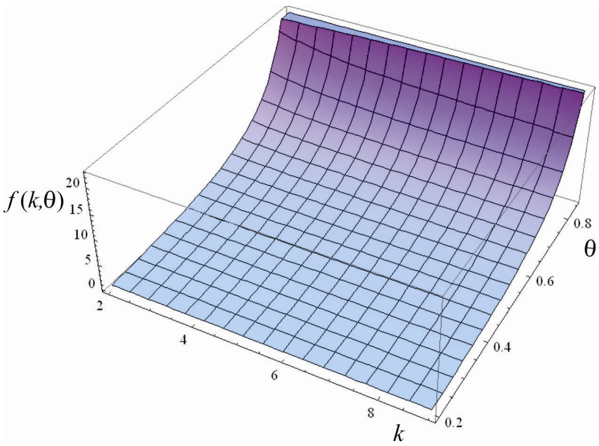

Figure 1. The graphic of the function $f(k,\theta)$ for $k \in \{2,3,4,\ldots,9\}$ and $\theta \in [0.2,0.85]$

Using the expression (8), related numerical results are given in Tables 1 and 2.

Table 1. The values of $f(k,\theta)$ for even values k = 2,4,6,8 and $\theta \in$ {0.25, 0.45, 0.65, .085}

	$f(k,\theta)$			
θ	k =2	k = 4	k = 6	k = 8
0.25	0.7083	0.6736	0.6620	0.6563
0.45	1.9543	1.7786	1.7200	1.6908
0.65	5.6975	4.9645	4.7202	4.5981
0.85	33.0083	27.7903	26.0509	25.181

Figure 1, Table 1 and Table 2 show that for any considered fixed k (and probably, also for all k = 2,3,4,…) the function $\theta \to f(k,\theta)$ increases on the segment [0.25,0.85] (and probably, also on the whole interval (0,1)). On the other hand, it is proved in Proposition of the previous section that for any fixed $\theta \in (0,1)$ the sequence $k \to f(k,\theta)$ decreases with respect to k (k = 2,3,4,…). Obviously, the same assertions are also true for the function $R(k,\theta) = r + f(k,\theta)$ defined by (8) which expresses the related values of specific cost ratio.

Finally, for r = 0.4 the graphic of the function $R(k,\theta)/R(k+1,\theta)$ in two variables $k \in \{2,3,4,\ldots 9\}$ and $\theta \in [0.25, 0.85]$ is presented in Figure 2. Related numerical results involving odd values k = 3,5,7,9 are given in Table 3.

Table 2. The values of $f(k,\theta)$ for odd values k = 3,5,7,9 and $\theta \in$ {0.25, 0.45, 0.65, 0.85}

	$f(k,\theta)$			
θ	k = 3	k = 5	k = 7	k = 9
0.25	0.6852	0.6667	0.6587	0.6543
0.45	1.8372	1.7435	1.7033	1.6810
0.65	5.2089	4.8180	4.6504	4.5574
0.85	29.5296	26.7467	25.5540	24.8914

Table 3. The values of $R(k,\theta)/R(k+1,\theta)$ with r = 0.4 for odd values k = 3,5,7,9

	$R(k,\theta)/R(k+1,\theta)$			
θ	k = 3	k = 5	k = 7	k = 9
0.25	1.0108	1.0044	1.0024	1.0015
0.45	1.0269	1.0111	1.0060	1.0038
0.65	1.0455	1.0191	1.0105	1.0066
0.85	1.0617	1.0263	1.0146	1.0093

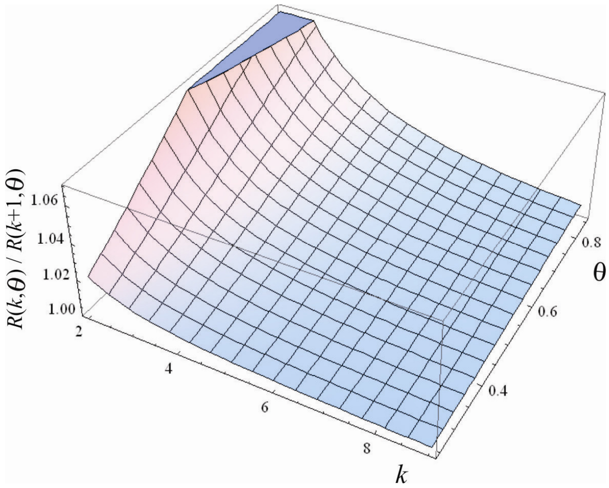

Figure 2. The graphic of the function $R(k,\theta)/R(k+1,\theta)$ with r = 0.4 for $k \in \{2,3,4,\ldots 9\}$ and $\theta \in [0.25, 0.85]$

5. CONCLUSION

In this paper, it is firstly derived the expression for the specific cost ratio in the dependence of shape parameter k and the traffic intensity θ for the port queuing model described as a single-server queue $M/E_k/1$ with one berth. Applying this expression, some theoretical, numerical and graphical results for the ted specific cost ratio involving different pairs of values k and θ are presented. The obtained analytic expression for specific cost ratio also gives the possibilities of discussing and comparing the values of specific cost ratio of various $M/E_k/1$ queues with specific cost ratio of the port queueing models investigated in earlier authors' papers. Moreover, the proposed analytical results would be useful for further development of this research field. In particular, this is closely related to the problem of minimization of specific cost ratio.

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МОДЕЛИРАЊЕ СПЕЦИФИЧНОГ ОДНОСА ТРОШКОВА У ЛУЦИ СА $M/E_k/1$ МОДЕЛОМ ТЕОРИЈЕ РЕДОВА ЧЕКАЊА

Б. Драговић, Р. Мештровић, Н. Зрнић,

Д. Драгојевић

Појам специфиичног односа трошкова односи се на разне типове трошкова брода и луке моделиране као систем реда чекања Користећи познату општу формулу за специфични однос трошкова (означен са R), изводимо одговарајући израз за R у односу na $M/E_k/1$ модел реда чекања, где је E_k (k = 2,3,...) Ерлангова k - расподела вероватноће. Овај израз нам омогућава да добијемо теоријски резултат који се може применити за одређивање оптималних вредности параметра облика k од E_k уз дата ограничења у односу на друге перформансе разматраног лучког модела теорије редова чекања. Такође су представљени одговарајући нумерички и графички резултати. Добијени резултати би могли бити корисни за будућа истраживања разматране проблематике.