A Study based on Stress-Strain Transfer Ratio Calculation using Halpin-Tsai and MROM Material Model for Limit Elastic Analysis of Metal Matrix FG Rotating Disk

Royal Madan

Research scholar
NIT Raipur
Mechanical Engineering
India

Shubhankar Bhowmick

Associate Professor
NIT Raipur
Mechanical Engineering
India

Kashinath Saha

Associate Professor
Jadavpur University
Faculty of Mechanical Engineering
India

Limit elastic analysis of a functionally graded (FG) rotating disk with material grading based on modified rule of mixture (MROM) is reported in the present study. In MROM, stress to strain transfer ratio (q) is an unknown parameter which restricts the application of MROM as yield strength estimation of a material depends on this ratio. Till now the determination of stress to strain transfer ratio, which varies with size, shape, manufacturing processes and composition, is reported by means of experiments. In current work, the effective Young's modulus is calculated by two means i.e. MROM and Halpin-Tsai. Later stress to strain transfer ratio is quantified using MROM using inverse approach and then effective yield stress variation of FGM has been calculated. Different combinations of metal matrix ceramic reinforced FG material were selected. Metals having high strength to weight ratio were combined with non-oxide ceramics of low density. Variational formulation method has been employed to solve the elasticity problem of the rotating disk taking radial displacement field as unknown variable and best material combination is proposed on the basis of maximum limit elastic speed obtained.

Keywords: Stress to strain transfer ratio, Modified rule of mixture(MROM) Limit analysis,Effective Young's modulus,Effective yield stress

1. INTRODUCTION

Rotating disk finds its applications in impeller, grinding wheel, flywheel, rotor, etc. Functionally graded materials are now very popular among industries because the components produced possess properties like low weight, high temperature resistance, corrosion resistance and high toughness. Depending upon the applications and operating temperature mainly metal-ceramic based FGM is used. The potential of a ceramic being in possession of high fracture toughness, high strength, low density, low thermal expansion, high temperature capability and oxidation resistance makes ceramics a suitable candidate as a reinforcement material. Metal matrix composites, being low weight and high strength members, when combined with ceramic, give good weight and cost savings. Potential applications include turbine blade, aerospace and internal combustion engines to name a few. FGM's can be fabricated using two approaches; step wise and continuously graded type. In layered type, multilayer material interface is present and in continuously graded type, the composition changes continuously with position [1]. Continuously graded variation is preferred as it optimizes the stress due to abrupt changes at the interface of graded material. Different materials used possess different thermo-mechanical properties like thermal expansion, which leads to stress concentration in a component [2], such effects may delaminate the component at the interface [3].

Disk performance can be improved by changing the geometry or by changing the material. In [4], investigation of FG disk of varying profiles such as linearly varying, converging and diverging profiles is reported. The results report less stresses in converging disk as compared to other profiles at same angular speed. To improve the performance further, like in rotating disks which require low weight and high strength at elevated temperatures different metals and ceramics combination is studied. Both aluminium and magnesium has high strength to weight ratio. Titanium, apart from having high strength to weight ratio, can maintain strength at elevated temperatures. Due to wettability issues of ceramics a maximum of 40 % of ceramic can only be added to form metal-ceramic FGM's. Non-oxide ceramics show improved strength and creep resistance compared to oxides, so, only non-oxides ceramics are analysed [5].

To identify effective modulus of FGM many methods are available. In [6], three averaging methods; the linear rule of mixture, MROM and the Wakashima–Tsukamoto estimate are reported wherein using the

finite-element approach, stress-strain transfer ratio, q is reported as 500 Gpa. Later in [7], considering the same value of q, volume fraction optimization on Ni–Al$_2$O$_3$ is performed to minimize thermal stresses using penalty-function and golden section method. Effective Young's modulus and Poisson's ratio of Ni/MgO and Ni$_3$Al/TiC functionally graded material was performed by [8] using Voronoi cell finite element method. Extended Mori–Tanaka method is reported by [9] to calculate effective modulus of a composite containing fractured particles.

Calculation of effective mechanical properties like Young's modulus, density, and yield strength is vital to calculate the displacement, strains and stresses effectively. In [10], effective modulus, yield strength and the effective strain-hardening coefficient of a metal-ceramic FGM using inverse analysis and indentation approach is presented. The effective property contains stress to strain transfer ratio as an unknown variable hence restrains its applications. The other method is a rule of mixture which is mostly used but it is found to be inaccurate in case of particulate composites as proved by [11] in which authors performed experiments on Al/SiC FGM's to calculate Young's modulus and then identified q from MROM relations given by [10].

The Young's modulus of a material changes with manufacturing technique used; size, shape of the particle and material composition. In current study effective Young's modulus is also calculated using Halpin-Tsai [12] method, and their comparison reveals related results when compared with MROM. The benefit of Halpin-Tsai is that the only variable associated is the shape of the particle which in present study is assumed as circular or square shaped whose aspect ratio is unity. Further MROM and Halpin-Tsai results were compared to identify stress to strain transfer ratio for further yield stress estimation of FGM's. For power law property variation, limit elastic analysis for different grading indices and aspect ratio has been reported by [15]. In one of the research [16], limit elastic analysis for different aspect ratio using modified rule of mixture is presented.

Till now no study has been carried out where an alternative way to calculate stress-strain transfer ratio has been proposed. The present article addresses the issue and establishes the material property calculation based on proposed hypothesis over an application of a most common industrial component, i.e. rotating disk. Effective yield stress variation in functionally graded rotating disk is also plotted to identify yield locations. Since the yield stress also varies along the disk radius, at a certain speed, at whatever location the induced stresses due to centrifugal loading of the disk becomes equal to the yield stress, that location is identified as yield location. Limit elastic results are also presented based on effective modulus and effective yield stress for different metal matrix-ceramic reinforced FGM combinations.

2. MATERIAL GRADING

2.1 Volume fraction: Volume fraction of ceramic and metal is quantified using Eq. (1) and Eq. (2) and shown in Figure 1. V_m and V_c are the volume fraction of metal and ceramic and r is the normalized radius of disk.

$$\begin{aligned} Vc = 0, \ & \xi \leq 0.2 \\ = 0.1, \ & 0.2 < \xi \leq 0.4 \\ = 0.2, \ & 0.4 < \xi \leq 0.6 \\ = 0.3, \ & 0.6 < \xi \leq 0.8 \\ = 0.4, \ & 0.8 < \xi \leq 1.0 \end{aligned} \tag{1}$$

(here ξ = (r-a)/(b-a))

$$V_m + V_c = 1 \tag{2}$$

2.2 Mass density: Density of FGM is calculated using rule of mixture given by Eq.3.

$$\rho_f = \rho_m V_m + \rho_c V_c \tag{3}$$

2.3 Young's modulus of Elasticity: When load is applied along fibre direction strains in each phase will be the same $(\varepsilon_c = \varepsilon_m)$ and stresses sum up as per Voigt model. Stresses will be same $(\sigma_c = \sigma_m)$ when transverse load is applied and strains get summed according to Reuss model. In case of particulate composites, neither Voigt nor Reuss can be used and hence both are combined to obtain modified rule of mixture given by Eq. 4, a detailed derivation of the same can be seen in [16].

The Young's modulus of metal and ceramic is taken from Table 1 and 2. To implement MROM, stress to strain transfer ratio (q) must be known which can be quantified experimentally. But to identify analytically a stress to strain transfer ratio for any material, Halpin-Tsai equation [12] given by (Eq. 6) is used and their results is compared with MROM for Al/SiC composites whose experimental data is known. Fig. 2 shows a slight variation in the Young's modulus values of both the methods used.
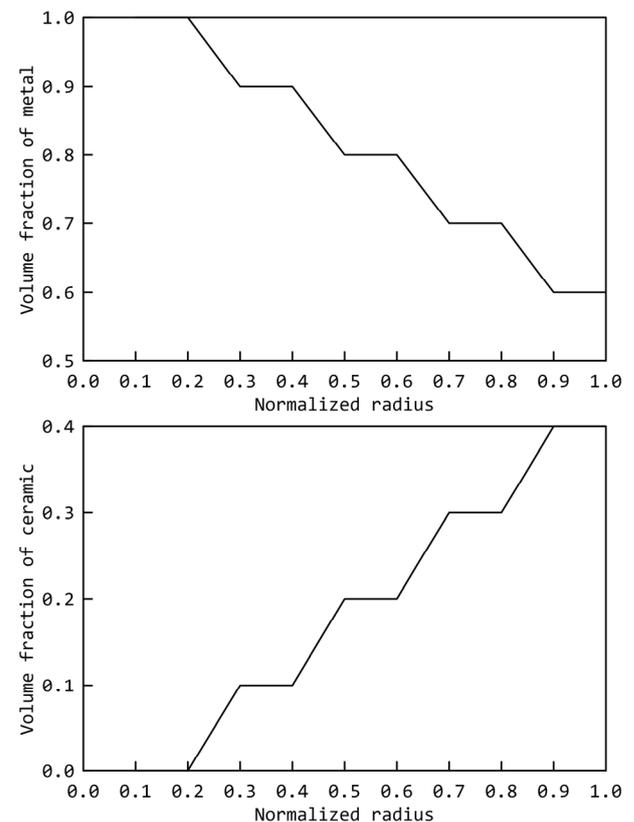
Figure 1. Volume fraction variation of (a) Metal (b) Ceramic

Table 1. Material property (at 1090 ^{0}C) of Metal [13]

	Material	E (GPa)	Density (kg/m^3)	Yield strength
1	Aluminium	67	2.643	270 MPa
2	Titanium	116	4.507	350 MPa
3	Magnesium	45	1.73	160 MPa

Table 2. Material property (at 1090 ^{0}C) of Ceramic [14]

	Material	E (GPa)	Density (kg/m^3)
1	SiC	324	3.21
2	Si$_3$N$_4$	317	3.31
3	TiB$_2$	414	4.50
4	AlN	310	3.26
5	TiC	448	4.94
6	ZrC	385	6.56

$$E_f = \frac{V_c E_c + V_m E_m R}{V_c + V_m R} \tag{4}$$

where

$$R = \frac{\overline{q}}{\overline{q} + \frac{E_m}{E_c}} \quad \text{and} \quad \overline{q} = \frac{q}{E_C}$$

The stress to strain transfer ratio can be written as [11]

$$q = -\frac{(\sigma_c - \sigma_m)}{(\varepsilon_c - \varepsilon_m)} \tag{5}$$

where $q = 0$, recovers Reuss model and $q = \infty$, recovers Voigt model.

$$\frac{E_T}{E_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f} \tag{6}$$

where $\eta = \dfrac{E_f / E_m - 1}{E_f / E_m + \xi}$ and $\xi = 2\dfrac{a}{b}$

2.4 Yield stress distribution: Yield stress distribution for FG is calculated using Eq. 7 for volume fraction variation given by Eq.1.

$$\sigma_{y_f} = \sigma_{ym}\left(V_m + \frac{E_c V_c}{E_m R}\right) \tag{7}$$

3. MATHEMATICAL FORMULATION

A uniform disk of thickness, h_0, inner radius a and outer radius b, respectively is considered for the study. The disk rotates at an angular speed, ω as shown in Figure 3, and due to centrifugal loading, radial and tangential, stresses and strains are induced. Plane stress condition is assumed. The strain energy and external work done due to centrifugal loading is calculated, and then by using constitutive and strain displacement relations, the following equation is obtained [16].

$$\delta\left[\frac{\pi}{1-\mu^2}\int_a^b\left\{E(r)\frac{u^2}{r} + E(r)2\mu u\frac{du}{dr} + E(r)r\left(\frac{du}{dr}\right)^2\right\}hdr\right. \\ \left. -2\pi\omega^2\int_a^b\rho(r)r^2uhdr\right] = 0 \tag{8}$$
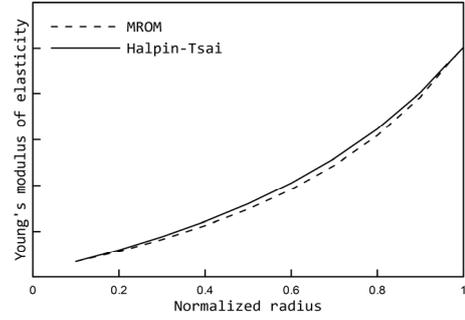
Figure 2. Comparison of Halpin-Tsai equation with MROM

SECTION A-A
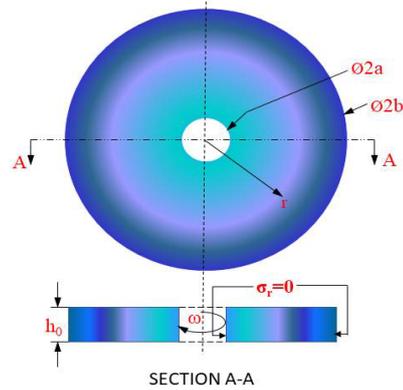
Figure 3. Uniform thickness FG rotating disk

Substituting normalized coordinate, as $\xi = (r-a)/(b-a)$, in Eq. (8) and taking $\overline{r} = (b-a)$, the following is obtained.

$$\delta\left[\frac{\pi}{1-\mu^2}\int_a^b E(r)\left\{\frac{u^2}{\overline{r}\xi+a} + \frac{2\mu u}{\overline{r}}\frac{du}{d\xi} + \frac{\overline{r}\xi+a}{\overline{r}^2}\left(\frac{du}{d\xi}\right)^2\right\}hd\xi \\ -2\pi\omega^2\overline{r}\int_a^b\rho(r)(\overline{r}\xi+a)^2uhd\xi\right] = 0 \tag{9}$$

Linear polynomial functions was assumed for *u* in Eq. (10), [17]

$$u(\xi) = \sum c_i\varphi_i, \quad i = 1,2,3,\ldots\ldots,n_f \tag{10}$$

Gram–Schmidt orthogonalization scheme is employed to identify higher order orthogonal functions. The start functions satisfy traction free boundary condition at the inner and outer radius. The start function thus obtained can be written as;

$$\varphi_O(r) = \frac{\omega^2 r(3+\mu)}{8} \cdot \\ \cdot\left[\frac{\rho(r)}{E(r)}\left\{(b^2+a^2)(1-\mu) - \frac{1+3\mu}{3+\mu}r^2 + \frac{a^2b^2}{r^2}(1+\mu)\right\}\right] \tag{11}$$

Upon substituting the governing equation is obtained in algebraic form:

$$\delta\left[\frac{\pi}{1-\mu^2}\int_0^1\left\{E(r)\left(\frac{(\sum c_i\varphi)^2}{\overline{r}\xi+a} + \frac{2\mu}{\overline{r}}\left[\sum c_i\varphi\frac{d(\sum c_i\varphi_i)}{d\xi}\right] \\ +\frac{\overline{r}\xi+a}{r^2}\left(\frac{d(\sum c_i\varphi_i)}{d\xi}\right)^2\right)\right\}hd\xi \\ -2\pi\omega^2\int_0^1\rho(r)\left\{(\overline{r}\xi+a)^2\sum c_i\varphi\right\}hd\xi\right] = 0 \tag{12}$$

In Eq. 12, the operator, δ is replaced by $\dfrac{\delta}{\delta c_j}$, $j = 1,2,3,4....n_f$.

Using Galerkin's error minimization principle; the following set of algebraic equations is obtained;

$$\frac{\bar{r}}{1-\mu^2}\sum_{i=1}^{n}\sum_{j=1}^{n}c_i\int_0^l E(r)\left\{\frac{\varphi_i\varphi_j}{\bar{r}\xi+a}+\left(\frac{\mu}{r}\left(\begin{array}{c}\varphi_i'\varphi_j\\+\varphi_i\varphi_j'\end{array}\right)\right)+\left(\frac{\bar{r}\xi+a}{r^2}\right)\left(\varphi_i'\varphi_j'\right)\right\}hd\xi \quad (13)$$

$$=\omega^2 b^3\sum_{i=1}^{n}\int_0^l\rho(r)\left\{\left(\bar{r}\xi+a\right)^2\varphi_i\right\}hd\xi$$

In matrix notation, Eq. 13 can be expressed as {**c**} = [**K**]$^{-1}${**R**} using in-house FORTRAN code. Further the displacements, strains and stresses in a disk are obtained. Finally the von-Mises stress is compared with the yield stress at each quadrature point along the disk radius to obtain limit elastic speed and location of yielding of FG rotating disk.

4. RESULTS AND DISCUSSION

The Young's modulus of FGM is calculated using Halpin - Tsai method, density is calculated based on rule of mixture and volume fraction variation is taken considering wettability issues. The normalized variable is taken as:

$$\bar{u} = \frac{u\,E_0}{b\,\sigma_{y0}},\quad \bar{\varepsilon} = \frac{\varepsilon E_0}{\sigma_{y0}},\quad \bar{\sigma}=\frac{\sigma}{\sigma_{y0}},\quad \bar{\sigma}_y = \frac{\sigma_y(r)}{\sigma_{y0}}$$

where, E_o, ρ_o, σ_{yo} are the elasticity modulus, density and yield stress at base (metal) and E_f, ρ_f, σ_{yf}, are the respective material properties at any radius r.

The limit elastic speed has not been normalized in the study as different material has different density and modulus values. As a consequence the results report dimensional limit angular speed. The effective modulus obtained from Halpin Tsai is used to calculate stress to strain transfer ratio *(q)* for different metal-ceramic combinations which is yet to be made available by any researcher prior to this report. Later these *(q)* values are substituted in Eq.7 which is used to calculate effective yield stress variation of FG rotating disk. The benefit of using Halpin-Tsai is that it takes care of shape of particle as well. The effective yield stress formulation used is an empirical relation and is a well-established relation which depends upon individual composition of graded materials. The other methods like the rule of mixture and power law seems less promising. Also one must remember that to perform limit elastic analysis, yield stress estimation at each quadrature point in the structural element is important to establish the limit load and location of yield initiation with in the structure.

Metal matrix composite, with different ceramic reinforcement, is considered for analysis. Depending upon the rotating disk applications three different metals having low weight to high strength ratio are selected (Al, Ti and Mg). The metals are combined with different ceramics (SiC, Si$_3$N$_4$, TiB$_2$, AIN, TiC, ZrC) and as a result several metal-ceramic combinations are investigated. It can be seen from Table 3, that different metal-ceramic combinations yield different limit elastic speeds. This helps in establishing the best combinations of graded material on the basis of obtained limit elastic angular speed. From Table 3, it is observed that the maximum limit speed obtained is for Al/SiC and Al/Si$_3$N$_4$, and minimum with Ti/ZrC. Similar analysis can be performed for any functionally graded material combinations wheather it is metal-ceramic, metal-metal or ceramic –ceramic. Figure 4 shows von-Mises stress and yield stress variation for different material combination for (*a/b*=0.1) along the disk radius. The point where both yield stress and von-Mises coincides defines the location of yield initiation and the corresponding speed is defined as the limit elastic speed of disk.

Table 3. Shows Limit speed and stress to strain transfer ratio for selected material combinations

	Ceramic	Metal	Stress to strain transfer ratio(*q*)	Angular speed in rad/sec
1		Al	0.41	372.78
2	SiC	Ti	0.70	338.36
3		Mg	0.27	333.81
4		Al	0.36	372.78
5	Si$_3$N$_4$	Ti	0.72	336.86
6		Mg	0.28	331.47
7		Al	0.32	355.24
8	TiB$_2$	Ti	0.55	329.10
9		Mg	0.21	312.27
10		Al	0.42	371.07
11	AlN	Ti	0.73	337.02
12		Mg	0.29	332.20
13		Al	0.30	349.80
14	TiC	Ti	0.51	326.15
15		Mg	0.20	305.65
16		Al	0.34	326.33
17	ZrC	Ti	0.60	309.23
18		Mg	0.23	281.49

Metals with Si₃N₄

(b) Metal with SiC

(c) Metals with TiB₂

(d) Metals with AlN

(e) Metals with TiC

(f) Metal with ZrC
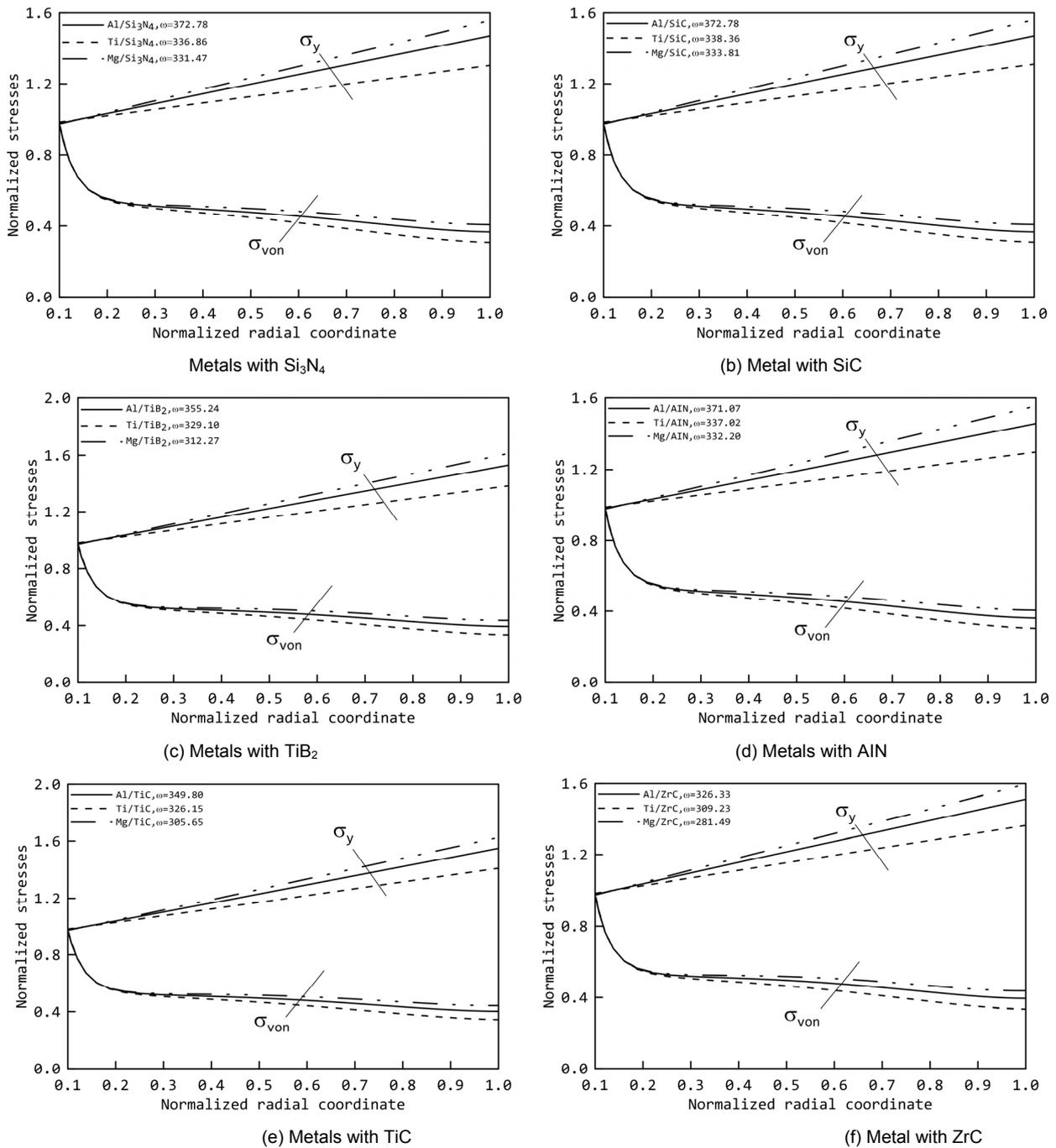
Figure 4 (a-f). Stress distribution in metal matrix ceramic reinforced FG disk

For all material combinations studied in the present article, the locations of yielding is obtained at the root of FG disk. However, this may not always happen as depending on the additional loading or disk geometry,

Metal matrix composite, with different ceramic reinforcement, is considered for analysis. Depending upon the rotating disk applications three different metals having low weight to high strength ratio are selected (Al, Ti and Mg). The metals are combined with different ceramics (SiC, Si₃N₄, TiB₂, AlN, TiC, ZrC) and as a result several metal-ceramic combinations are investigated. It can be seen from Table 3, that different metal-ceramic combinations yield different limit elastic speeds. This helps in establishing the best combinations of graded material on the basis of obtained limit elastic angular speed. From Table 3, it is observed that the

the location of yielding may appear at positions away from root of the disk. In such cases, any further incre- ment in the rotational speed will cause the com- mencement of bi-directional propagation of yield front. maximum limit speed obtained is for Al/SiC and Al/Si₃N₄, and minimum with Ti/ZrC. Similar analysis can be performed for any functionally graded material combi- nations wheather it is metal-ceramic, metal-metal or ceramic–ceramic. Figure 4 shows von-Mises stress and yield stress variation for different material combination for (*a/b*=0.1) along the disk radius. The point where both yield stress and von-Mises coincides defines the location of yield initiation and the corresponding speed is defined as the limit elastic speed of disk. For all material combinations studied in the present article, the locations of yielding is obtained at the root of FG disk. However,

this may not always happen as depending on the additional loading or disk geometry, the location of yielding may appear at positions away from root of the disk. In such cases, any further increment in the rotational speed will cause the commencement of bi-directional propagation of yield front.

5. CONCLUSION

The present study reveals an alternative way to calculate stress to strain transfer ratio for any metal-ceramic FG combinations. It is established that stress-strain transfer ratio will not change considerably with composition of reinforced material and hence an average value can be considered for analysis purpose. Other than this it varies with; size, shape of particle and manufacturing processes. Limit elastic analysis of a high speed rotating FG disk with yield stress variation derived on the basis of proposed stress-strain transfer ratio was performed based on effective yield stress and effective Young's modulus to identify suitable material combination. The present study can be extended to any new material combinations depending upon the applications. The location of yielding is identified from stress plot and for the present material combinations, is obtained at the root of the disk. The best material combination selection is done on the basis of maximum limit speed obtained which is the key essentials of rotating structures as increased limit elastic speed results in increased operating range of rotating structures such as flywheels, impellers etc.

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ПРОУЧАВАЊЕ РОТИРАЈУЋЕГ ДИСКА ОД ФУНКЦИОНАЛНО ГРАДИРАНОГ МАТЕРИЈАЛА СА МЕТАЛНОМ МАТРИЦОМ БАЗИРАНО НА ПРЕНОСНОМ ОДНОСУ НАПОНА И ДЕФОРМАЦИЈЕ ПРИМЕНОМ ХЕЛПИН-ЦАЈОВОГ И MROM МОДЕЛА ЗА АНАЛИЗУ ГРАНИЦЕ ЕЛАСТИЧНОСТИ

Р. Мадан, С. Бхоумик, К. Саха

Рад се бави анализом границе еластичности ротирајућег диска израђеног од функционално градираног материјала, са градацијом извршеном применом модификованог правила смеше (MROM). Код MROM-а преносни однос напона и деформације је изражен непознатим параметром који ограничава примену MROM-а, јер процена напона течења материјала зависи од овог односа. До сада је преносни однос напона и деформације, који иначе варира са димензијама и обликом, одређиван процесом производње и саставом материјала, као и експерименталним путем. У овом раду је израчунат

Јунгов модул применом Халпин Цајевог и MROM модела. Касније је примењен и инверзни метод и израчуната је варијација напона течења функционално градираног материјала. Изабране су различите комбинације материјала са металном матрицом ојачаног керамиком. Метали који имају добар однос чврстоће и тежине су комбиновани са неоксидном керамиком мале густине. Метод варијационе формулације је коришћен за решавање проблема еластичности ротирајућег диска, при чему је поље радијалног помераја узето за непознату променљиву а најбоља комбинација материјала је предложена на основу добијене максималне границе еластичности.